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## ► To cite this version:

Anne-Laure Basdevant, Arvind Singh. Rate of growth of a transient cookie random walk. 2007.  
hal-00135952

**HAL Id: hal-00135952**

**<https://hal.science/hal-00135952>**

Preprint submitted on 9 Mar 2007

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# Rate of growth of a transient cookie random walk

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## Abstract

We consider a one-dimensional transient cookie random walk. It is known from a previous paper [3] that a cookie random walk  $(X_n)$  has positive or zero speed according to some positive parameter  $\alpha > 1$  or  $\leq 1$ . In this article, we give the exact rate of growth of  $(X_n)$  in the zero speed regime, namely: for  $0 < \alpha < 1$ ,  $X_n/n^{\frac{\alpha+1}{2}}$  converges in law to a Mittag-Leffler distribution whereas for  $\alpha = 1$ ,  $X_n(\log n)/n$  converges in probability to some positive constant.

**Keywords.** Rates of transience; cookie or multi-excited random walk; branching process with migration

**A.M.S. Classification.** 60K35, 60J80, 60F05

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## 1 Introduction

Let us pick a strictly positive integer  $M$ . An  $M$ -cookie random walk (also called multi-excited random walk) is a walk on  $\mathbb{Z}$  which has a bias to the right upon its  $M$  first visits at a given site and evolves like a symmetric random walk afterwards. This model was introduced by Zerner [16] as a generalization, in the one-dimensional setting, of the model of the excited random walk studied by Benjamini and Wilson [4]. In this paper, we consider the case where the initial cookie environment is spatially homogeneous. Formally, let  $(\Omega, \mathbf{P})$  be some probability space and choose a vector  $\bar{p} = (p_1, \dots, p_M)$  such that  $p_i \in [\frac{1}{2}, 1)$  for all  $i = 1, \dots, M$ . We say that  $p_i$  represents the strength of the  $i^{\text{th}}$  cookie at a given site. Then, an  $(M, \bar{p})$ -cookie random walk  $(X_n, n \in \mathbb{N})$  is a nearest neighbour random walk, starting from 0, and with transition probabilities:

$$\mathbf{P}\{X_{n+1} = X_n + 1 - X_0, \dots, X_n\} = \begin{cases} p_j & \text{if } j = \#\{0 \leq i \leq n, X_i = X_n\} \leq M, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

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In particular, the future position  $X_{n+1}$  of the walk after time  $n$  depends on the whole trajectory  $X_0, X_1, \dots, X_n$ . Therefore,  $X$  is not, unless in degenerated cases, a Markov process. The cookie random walk is a rich stochastic model. Depending on the cookie environment  $(M, \bar{p})$ , the process can either be transient or recurrent. Precisely, Zerner [16] (who considered an even more general setting) proved, in our case, that if we define

$$\alpha = \alpha(M, \bar{p}) \stackrel{\text{def}}{=} \sum_{i=1}^M (2p_i - 1) - 1, \quad (1.1)$$

then

- if  $\alpha \leq 0$ , the cookie random walk is recurrent,
- if  $\alpha > 0$ , the cookie random walk is transient towards  $+\infty$ .

Thus, a 1-cookie random walk is always recurrent but, for two or more cookies, the walk can either be transient or recurrent. Zerner also proved that the limiting velocity of the walk is well defined. That is, there exists a deterministic constant  $v = v(M, \bar{p}) \geq 0$  such that

$$\lim_{n \rightarrow \infty} \frac{X_n}{n} = v \quad \text{almost surely.}$$

However, we may have  $v = 0$ . Indeed, when there are at most two cookies per site, Zerner proved that  $v$  is always zero. On the other hand, Mountford *et al.* [9] showed that it is possible to have  $v > 0$  if the number of cookies is large enough. In a previous paper [3], the authors showed that, in fact, the strict positivity of the speed depends on the position of  $\alpha$  with respect to 1:

- if  $\alpha \leq 1$ , then  $v = 0$ ,
- if  $\alpha > 1$ , then  $v > 0$ .

In particular, a positive speed may be obtained with just three cookies per site. The aim of this paper is to find the exact rate of growth of a transient cookie random walk in zero speed regime. In this perspective, numerical simulations of Antal and Redner [2] indicated that, for a transient 2-cookies random walk, the expectation of  $X_n$  is of order  $n^\nu$ , for some constant  $\nu \in (\frac{1}{2}, 1)$  depending on the strength of the cookies. We shall prove that, more generally,  $\nu = \frac{\alpha+1}{2}$ .

**Theorem 1.1.** *Let  $X$  be a  $(M, \bar{p})$ -cookie random walk and let  $\alpha$  be defined by (1.1). Then, when the walk is transient with zero speed, i.e. when  $0 < \alpha \leq 1$ ,*

1. *If  $\alpha < 1$ ,*

$$\frac{X_n}{n^{\frac{\alpha+1}{2}}} \xrightarrow[n \rightarrow \infty]{law} \mathcal{M}_{\frac{\alpha+1}{2}}$$

*where  $\mathcal{M}_{\frac{\alpha+1}{2}}$  denotes a Mittag-Leffler distribution with parameter  $\frac{\alpha+1}{2}$ .*

2. *If  $\alpha = 1$ , there exists a constant  $c > 0$  such that*

$$\frac{\log n}{n} X_n \xrightarrow[n \rightarrow \infty]{prob.} c.$$

*These results also hold with  $\sup_{i \leq n} X_i$  and  $\inf_{i \geq n} X_i$  in place of  $X_n$ .*

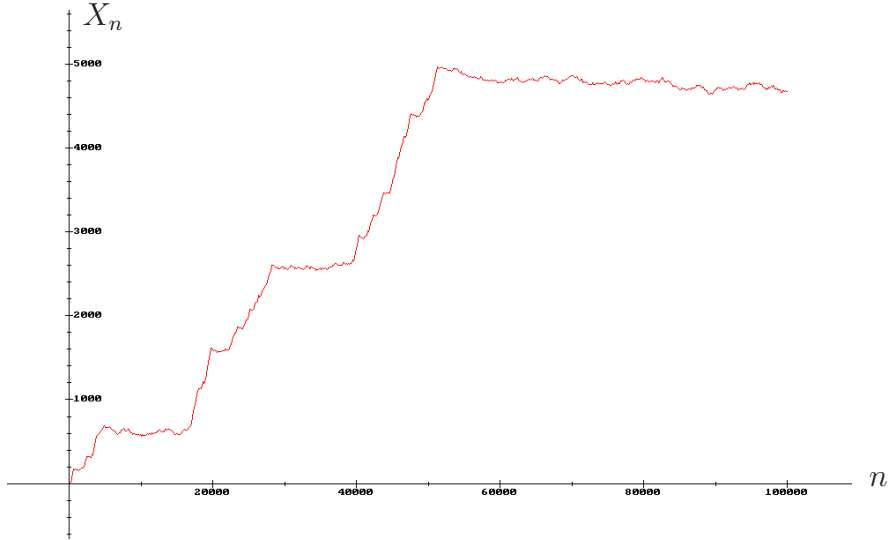


Figure 1: Simulation of the 100000 first steps of a cookie random walk with  $M = 3$  and  $p_1 = p_2 = p_3 = \frac{3}{4}$  (*i.e.*  $\alpha = \frac{1}{2}$  and  $\nu = \frac{3}{4}$ ).

This theorem bears many likenesses to the famous result of Kesten-Kozlov-Spitzer [7] concerning the rate of transience of a one-dimensional random walk in random environment. Indeed, following the method initiated in [3], we can reduce the study of the walk to that of an auxiliary Markov process  $Z$ . In our setting,  $Z$  is a branching process with migration. By comparison, Kesten *et al.* obtained the rates of transience of the random walk in random environment via the study of an associated branching process in random environment. However, the process  $Z$  considered here and the process introduced in [7] have quite dissimilar behaviours and the methods used for their study are fairly different.

Let us also note that, as  $\alpha$  tends to zero, the rate of growth  $n^{(1+\alpha)/2}$  tends to  $\sqrt{n}$ . This suggests that, when the cookie walk is recurrent (*i.e.*  $-1 < \alpha \leq 0$ ), its growth should not be much larger than that of a simple symmetric random walk. In fact, we believe that, in the recurrent setting,  $\sup_{i \leq n} X_i$  should be of order  $l(n)\sqrt{n}$  for some slowly varying function  $l$ .

The remainder of this paper is organized as follow. In the next section, we recall the construction of the associated process  $Z$  described in [3] as well as some important results concerning this process. In section 3, we study the tail distribution of the return time to zero of the process  $Z$ . Section 4 is devoted to estimating the tail distribution of the total progeny of the branching process over an excursion away from 0. The proof of this result is based on technical estimates whose proofs are given in section 5. Once all these results obtained, the proof of the main theorem is quite straightforward and is finally given in the last section.

## 2 The process $Z$

In the rest of this paper,  $X$  will denote an  $(M, \bar{p})$ -cookie random walk. We will also always assume that we are in the transient regime and that the speed of the walk is zero, that is

$$0 < \alpha \leq 1.$$

The proof of Theorem 1.1 is based on a careful study of the hitting times of the walk:

$$T_n \stackrel{\text{def}}{=} \inf\{k \geq 0, X_k = n\}.$$

We now introduce a Markov process  $Z$  closely connected with these hitting times. Indeed, we can summarize Proposition 2.2 and equation (4) of [3] as follows:

**Proposition 2.1.** *There exist a Markov process  $(Z_n, n \in \mathbb{N})$  starting from 0 and a sequence of random variables  $(K_n, n \geq 0)$  converging in law towards a finite random variable  $K$  such that, for each  $n$*

$$T_n \stackrel{\text{law}}{=} n + 2 \sum_{k=0}^n Z_k + K_n.$$

Therefore, a careful study of  $Z$  will enable us to obtain precise estimates on the distribution of the hitting times. In the rest of this section, we shall recall the construction of  $Z$  and some important results obtained in [3].

For each  $i = 1, 2, \dots$ , let  $B_i$  be a Bernoulli random variable with distribution

$$\mathbf{P}\{B_i = 1\} = 1 - \mathbf{P}\{B_i = 0\} = \begin{cases} p_i & \text{if } 1 \leq i \leq M, \\ \frac{1}{2} & \text{if } i > M. \end{cases}$$

We define the random variables  $A_0, A_1, \dots, A_{M-1}$  by

$$A_j \stackrel{\text{def}}{=} \#\{1 \leq i \leq k_j, B_i = 0\} \quad \text{where} \quad k_j \stackrel{\text{def}}{=} \inf \left( i \geq 1, \sum_{l=1}^i B_l = j + 1 \right).$$

Therefore,  $A_j$  represents the number of "failures" before having  $j + 1$  "successes" along the sequence of coin tossings  $(B_i)$ . It is to be noted that the random variables  $A_j$  admit some exponential moments:

$$\mathbf{E}[s^{A_j}] < \infty \quad \text{for all } s \in [0, 2). \quad (2.1)$$

According to Lemma 3.3 of [3], we also have

$$\mathbf{E}[A_{M-1}] = 2 \sum_{i=1}^M (1 - p_i) = M - 1 - \alpha. \quad (2.2)$$

Let  $(\xi_i, i \in \mathbb{N}^*)$  be a sequence of i.i.d. geometric random variables with parameter  $\frac{1}{2}$  (*i.e.* with mean 1), independent of the  $A_j$ . The process  $Z$  mentioned above is a Markov process with transition probabilities given by

$$\mathbf{P}\{Z_{n+1} = j - Z_n = i\} = \mathbf{P}\left\{ \mathbb{1}_{\{i \leq M-1\}} A_i + \mathbb{1}_{\{i > M-1\}} \left( A_{M-1} + \sum_{k=1}^{i-M+1} \xi_k \right) = j \right\}. \quad (2.3)$$

As usual, we will use the notation  $\mathbf{P}_x$  to describe the law of the process starting from  $x \in \mathbb{N}$  and  $\mathbf{E}_x$  the associated expectation, with the conventions  $\mathbf{P} = \mathbf{P}_0$  and  $\mathbf{E} = \mathbf{E}_0$ . Let us notice that  $Z$  may be interpreted as a branching process with random migration, that is, a branching process which allows both immigration and emigration components.

- If  $Z_n = i \in \{M, M+1, \dots\}$ , then  $Z_{n+1}$  has the law of  $\sum_{k=1}^{i-M+1} \xi_k + A_{M-1}$ , *i.e.*  $M-1$  particles emigrate from the system and the remaining particles reproduce according to a geometrical law with parameter  $\frac{1}{2}$  and there is also an immigration of  $A_{M-1}$  new particles.
- If  $Z_n = i \in \{0, \dots, M-1\}$ , then  $Z_{n+1}$  has the same law as  $A_i$ , *i.e.* all the  $i$  particles emigrate the system and  $A_i$  new particles immigrate.

Since we assume that the cookie vector  $\bar{p}$  is such that  $p_i < 1$  for all  $i$ , the process  $Z$  is an irreducible Markov process. More precisely,

$$\mathbf{P}_x\{Z_1 = y\} > 0 \quad \text{for all } x, y \in \mathbb{N}.$$

From the construction of the random variables  $A_i$ , we have  $A_0 \leq A_1 \leq \dots \leq A_{M-1}$ . This fact easily implies that, for any  $x \leq y$ , the process  $Z$  under  $\mathbf{P}_x$  (starting from  $x$ ) is stochastically dominated by  $Z$  under  $\mathbf{P}_y$  (starting from  $y$ ). Let us also note that, for any  $k \geq M-1$ ,

$$\mathbf{E}[Z_{n+1} - Z_n \mid Z_n = k] = \mathbf{E}[A_{M-1}] - M + 1 = -\alpha. \quad (2.4)$$

This quantity is negative and we say that emigration dominates immigration. In view of (2.4), a simple martingale argument shows that  $Z$  is recurrent. More precisely, according to section 2 of [3], the process  $Z$  is, in fact, positive recurrent and thus converges in law, independently of its starting point, towards a random variable  $Z_\infty$  whose law is the unique invariant probability for  $Z$ . Moreover, according to Remark 3.7 of [3], the tail distribution of  $Z_\infty$  is regularly varying with index  $\alpha$ :

**Proposition 2.2.** *There exists a constant  $c > 0$  such that*

$$\mathbf{P}\{Z_\infty > x\} \underset{x \rightarrow \infty}{\sim} \begin{cases} c/x^\alpha & \text{if } \alpha \in (0, 1), \\ c \log x / x & \text{if } \alpha = 1. \end{cases}$$

Let now  $\sigma$  denote the first return time to 0 for the process  $Z$ ,

$$\sigma \stackrel{\text{def}}{=} \inf\{n \geq 1, Z_n = 0\}.$$

According to the classical expression of the invariant probability, for any non negative function  $f$ , we have

$$\mathbf{E}\left[\sum_{i=0}^{\sigma-1} f(Z_i)\right] = \mathbf{E}[\sigma]\mathbf{E}[f(Z_\infty)]. \quad (2.5)$$

In particular, we deduce the following corollary which will be found very useful:

**Corollary 2.3.** *We have, for  $\beta \geq 0$ ,*

$$\mathbf{E}\left[\sum_{i=0}^{\sigma-1} Z_i^\beta\right] \begin{cases} < \infty & \text{if } \beta < \alpha, \\ = \infty & \text{if } \beta \geq \alpha. \end{cases}$$

### 3 The return time to zero

We have already stated that  $Z$  is an irreducible positive recurrent Markov chain, thus the return time  $\sigma$  to zero has finite expectation. The aim of this section is to strengthen this result by giving the asymptotic of the tail distribution of  $\sigma$ . Precisely, we will show that

**Proposition 3.1.** *For any initial starting point  $x \geq 1$ , there exists  $c = c(x) > 0$  such that*

$$\mathbf{P}_x\{\sigma > n\} \underset{n \rightarrow \infty}{\sim} \frac{c}{n^{\alpha+1}}.$$

Notice that we do not allow the starting point  $x$  to be 0. In fact, this assumption could be dropped but it would unnecessarily complicate the proof of the proposition which is technical enough already. Yet, we have already mentioned that  $Z$  starting from 0 is stochastically dominated by  $Z$  starting from 1, thus  $\mathbf{P}\{\sigma > n\} \leq \mathbf{P}_1\{\sigma > n\}$ . We also have  $\mathbf{P}\{\sigma > n\} \geq \mathbf{P}\{Z_1 = 1\}\mathbf{P}_1\{\sigma > n - 1\}$ . Therefore, we deduce that

$$\frac{c_1}{n^{\alpha+1}} \leq \mathbf{P}\{\sigma > n\} \leq \frac{c_2}{n^{\alpha+1}}$$

where  $c_1$  and  $c_2$  are two strictly positive constants. In particular, we obtain the following corollary which will be sufficient for our needs.

**Corollary 3.2.** *We have*

$$\mathbf{E}[\sigma^\beta] \begin{cases} < \infty & \text{if } \beta < \alpha + 1, \\ = \infty & \text{if } \beta \geq \alpha + 1. \end{cases} \quad (3.1)$$

The method used in the proof of the proposition is classical and based on the study of probability generating functions. Proposition 3.1 was first proved by Vatutin [10] who considered a branching process with exactly one emigrant at each generation. This result was later generalized for branching processes with more than one emigrant by Vinokurov [12] and also by Kaverin [6]. However, in our setting, we deal with a branching process with migration, that is, where both immigration and emigration are allowed. More recently, Yanev and Yanev proved similar results for such a class of processes, under the assumption that, either there is at most one emigrant per generation [14] or that immigration dominates emigration [13] (in our setting, this would correspond to the case  $\alpha < 0$ ).

For the process  $Z$ , the emigration component dominates the immigration component and this leads to some additional technical difficulties. Although there is a vast literature on the subject (see the authoritative survey of Vatutin and Zubkov [11] for additional references), we did not find a proof of Proposition 3.1 in our setting. We shall therefore provide here a complete argument but we invite the reader to look in the references mentioned above for additional details.

Recall the definition of the random variables  $A_i$  and  $\xi_i$  defined in section 2. We introduce, for  $s \in [0, 1]$ ,

$$\begin{aligned} F(s) &\stackrel{\text{def}}{=} \mathbf{E}[s^{\xi_1}] = \frac{1}{2-s}, \\ \delta(s) &\stackrel{\text{def}}{=} (2-s)^{M-1} \mathbf{E}[s^{A_{M-1}}], \\ H_k(s) &\stackrel{\text{def}}{=} (2-s)^{M-1-k} \mathbf{E}[s^{A_{M-1}}] - \mathbf{E}[s^{A_k}] \quad \text{for } 1 \leq k \leq M-2. \end{aligned}$$

Let  $F_j(s) \stackrel{\text{def}}{=} F \circ \dots \circ F(s)$  stand for the  $j$ -fold of  $F$  (with the convention  $F_0 = \text{Id}$ ). We also define by induction

$$\begin{cases} \gamma_0(s) \stackrel{\text{def}}{=} 1, \\ \gamma_{n+1}(s) \stackrel{\text{def}}{=} \delta(F_n(s))\gamma_n(s). \end{cases}$$

We use the abbreviated notations  $F_j \stackrel{\text{def}}{=} F_j(0)$ ,  $\gamma_n \stackrel{\text{def}}{=} \gamma_n(0)$ . We start with a simple lemma.

**Lemma 3.3.** (a)  $F_n = 1 - \frac{1}{n+1}$ .

(b)  $H_k(1-s) = -H'_k(1)s + \mathcal{O}(s^2)$  when  $s \rightarrow 0$  for all  $1 \leq k \leq M-2$ .

(c)  $\delta(1-s) = 1 + \alpha s + \mathcal{O}(s^2)$  when  $s \rightarrow 0$ .

(d)  $\gamma_n \sim_{\infty} c_3 n^{\alpha}$  with  $c_3 > 0$ .

*Proof.* Assertion (a) is straightforward. According to (2.1), the functions  $H_k$  are analytic on  $(0, 2)$  and (b) follows from a Taylor expansion near 1. Similarly, (c) follows from a Taylor expansion near 1 of the function  $\delta$  combined with (2.2). Finally,  $\gamma_n$  can be expressed in the form

$$\gamma_n = \prod_{j=0}^{n-1} \delta(F_j) \underset{n \rightarrow \infty}{\sim} \prod_{j=1}^n \left(1 + \frac{\alpha}{j}\right) \underset{n \rightarrow \infty}{\sim} c_3 n^{\alpha},$$

which yields (d). □

Let  $\tilde{Z}$  stand for the process  $Z$  absorbed at 0:

$$\tilde{Z}_n \stackrel{\text{def}}{=} Z_n \mathbb{1}_{\{n \leq \inf(k, Z_k=0)\}}.$$

We also define, for  $x \geq 1$  and  $s \in [0, 1]$ ,

$$\begin{aligned} J_x(s) &\stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \mathbf{P}_x\{\tilde{Z}_i \neq 0\} s^i, \\ G_{n,x}(s) &\stackrel{\text{def}}{=} \mathbf{E}_x[s^{\tilde{Z}_n}], \end{aligned} \tag{3.2}$$

and for  $1 \leq k \leq M-2$ ,

$$g_{k,x}(s) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} \mathbf{P}_x\{\tilde{Z}_i = k\} s^{i+1}.$$

**Lemma 3.4.** For any  $1 \leq k \leq M-2$ , we have

(a)  $\sup_{x \geq 1} g_{k,x}(1) < \infty$ .

(b) for all  $x \geq 1$ ,  $g'_{k,x}(1) < \infty$ .



*Proof.* The value  $g_{x,k}(1)$  represents the expected number of visits to site  $k$  before hitting 0 for the process  $Z$  starting from  $x$ . Thus, an easy application of the Markov property yields

$$g_{k,x}(1) = \frac{\mathbf{P}_x\{Z \text{ visits } k \text{ before } 0\}}{\mathbf{P}_k\{Z \text{ visits } 0 \text{ before returning to } k\}} < \frac{1}{\mathbf{P}_k\{Z_1 = 0\}} < \infty.$$

This proves (a). We now introduce the return times  $\sigma_k \stackrel{\text{def}}{=} \inf(n \geq 1, Z_n = k)$ . In view of the Markov property, we have

$$\begin{aligned} g'_{k,x}(1) &= g_{k,x}(1) + \mathbf{E}_x \left[ \sum_{n=1}^{\infty} n \mathbb{1}_{\{\tilde{Z}_n=k\}} \right] \\ &= g_{k,x}(1) + \sum_{i=1}^{\infty} \mathbf{P}_x\{\sigma_k = i, \sigma_k < \sigma\} \mathbf{E}_k \left[ \sum_{n=0}^{\infty} (i+n) \mathbb{1}_{\{\tilde{Z}_n=k\}} \right] \\ &= g_{k,x}(1) + \mathbf{E}_x[\sigma_k \mathbb{1}_{\{\sigma_k < \sigma\}}] g_{k,k}(1) + \mathbf{P}_x\{\sigma_k < \sigma\} \mathbf{E}_k \left[ \sum_{n=0}^{\infty} n \mathbb{1}_{\{\tilde{Z}_n=k\}} \right]. \end{aligned}$$

Since  $Z$  is a positive recurrent Markov process, we have  $\mathbf{E}_x[\sigma_k \mathbb{1}_{\{\sigma_k < \sigma\}}] \leq \mathbf{E}_x[\sigma] < \infty$ . Thus, it simply remains to show that  $\mathbf{E}_k \left[ \sum_{n=0}^{\infty} n \mathbb{1}_{\{\tilde{Z}_n=k\}} \right] < \infty$ . Using the Markov property, as above, but considering now the partial sums, we get, for any  $N \geq 1$ ,

$$\begin{aligned} \mathbf{E}_k \left[ \sum_{n=1}^N n \mathbb{1}_{\{\tilde{Z}_n=k\}} \right] &= \sum_{i=1}^N \mathbf{P}_k\{\sigma_k = i, \sigma_k < \sigma\} \mathbf{E}_k \left[ \sum_{n=0}^{N-i} (i+n) \mathbb{1}_{\{\tilde{Z}_n=k\}} \right] \\ &\leq \mathbf{E}_k[\sigma_k \mathbb{1}_{\{\sigma_k < \sigma\}}] g_{k,k}(1) + \mathbf{P}_k\{\sigma_k < \sigma\} \mathbf{E}_k \left[ \sum_{n=1}^N n \mathbb{1}_{\{\tilde{Z}_n=k\}} \right]. \end{aligned}$$

Since  $\mathbf{P}_k\{\sigma < \sigma_k\} \geq \mathbf{P}_k\{Z_1 = 0\} > 0$ , we deduce that

$$\mathbf{E}_k \left[ \sum_{n=1}^N n \mathbb{1}_{\{\tilde{Z}_n=k\}} \right] \leq \frac{\mathbf{E}_k[\sigma_k \mathbb{1}_{\{\sigma_k < \sigma\}}] g_{k,k}(1)}{\mathbf{P}_k\{\sigma < \sigma_k\}} < \infty.$$

and we conclude the proof letting  $N$  tend to  $+\infty$ . □

**Lemma 3.5.** *The function  $J_x$  defined by (3.2) may be expressed in the form*

$$J_x(s) = \hat{J}_x(s) + \sum_{k=1}^{M-2} \tilde{J}_{k,x}(s) \quad \text{for } s \in [0, 1),$$

where

$$\hat{J}_x(s) \stackrel{\text{def}}{=} \frac{\sum_{n=0}^{\infty} \gamma_n (1 - (F_n)^x) s^n}{(1-s) \sum_{n=0}^{\infty} \gamma_n s^n} \quad \text{and} \quad \tilde{J}_{k,x}(s) \stackrel{\text{def}}{=} \frac{g_{k,x}(s) \sum_{n=0}^{\infty} \gamma_n H_k(F_n) s^n}{(1-s) \sum_{n=0}^{\infty} \gamma_n s^n}.$$

*Proof.* From the definition (2.3) of the branching process  $Z$ , we get, for  $n \geq 0$ ,

$$\begin{aligned}
G_{n+1,x}(s) &= \mathbf{E}_x \left[ \mathbf{E}_{\tilde{Z}_n} [s^{\tilde{Z}_1}] \right] \\
&= \mathbf{P}_x \{ \tilde{Z}_n = 0 \} + \sum_{k=1}^{M-2} \mathbf{P}_x \{ \tilde{Z}_n = k \} \mathbf{E}[s^{A_k}] + \sum_{k=M-1}^{\infty} \mathbf{P}_x \{ \tilde{Z}_n = k \} \mathbf{E}[s^{\xi}]^{k-(M-1)} \mathbf{E}[s^{A_{M-1}}] \\
&= \left( 1 - \frac{\mathbf{E}[s^{A_{M-1}}]}{\mathbf{E}[s^{\xi}]^{M-1}} \right) \mathbf{P}_x \{ \tilde{Z}_n = 0 \} - \sum_{k=1}^{M-2} \mathbf{P}_x \{ \tilde{Z}_n = k \} H_k(s) + \frac{\mathbf{E}[s^{A_{M-1}}]}{\mathbf{E}[s^{\xi}]^{M-1}} \sum_{k=0}^{\infty} \mathbf{P}_x \{ \tilde{Z}_n = k \} \mathbf{E}[s^{\xi}]^k.
\end{aligned}$$

Since  $\mathbf{E}[s^{\xi}] = F(s)$  and  $G_{n,x}(0) = \mathbf{P}_x \{ \tilde{Z}_n = 0 \}$ , using the notation introduced in the beginning of the section, the last equality may be rewritten

$$G_{n+1,x}(s) = \delta(s) G_{n,x}(F(s)) + (1 - \delta(s)) G_{n,x}(0) - \sum_{k=1}^{M-2} \mathbf{P}_x \{ \tilde{Z}_n = k \} H_k(s).$$

Iterating this equation then setting  $s = 0$  and using the relation  $G_{0,x}(F_{n+1}) = (F_{n+1})^x$ , we deduce that, for any  $n \geq 0$ ,

$$G_{n+1,x}(0) = \sum_{i=0}^n (1 - \delta(F_i)) \gamma_i G_{n-i,x}(0) + \gamma_{n+1} (F_{n+1})^x - \sum_{k=1}^{M-2} \sum_{i=0}^n \mathbf{P}_x \{ \tilde{Z}_{n-i} = k \} \gamma_i H_k(F_i). \quad (3.3)$$

Notice also that  $\mathbf{P}_x \{ \tilde{Z}_n \neq 0 \} = 1 - G_{n,x}(0)$ . In view of (3.3) and making use of the relation  $(1 - \delta(F_i)) \gamma_i = \gamma_i - \gamma_{i+1}$ , we find, for all  $n \geq 0$  (with the convention  $\sum_0^{-1} = 0$ )

$$\begin{aligned}
\mathbf{P}_x \{ \tilde{Z}_n \neq 0 \} &= \gamma_n (1 - (F_n)^x) + \sum_{i=0}^{n-1} (\gamma_i - \gamma_{i+1}) \mathbf{P}_x \{ \tilde{Z}_{n-1-i} \neq 0 \} \\
&\quad + \sum_{k=1}^{M-2} \sum_{i=0}^{n-1} \mathbf{P}_x \{ \tilde{Z}_{n-1-i} = k \} \gamma_i H_k(F_i).
\end{aligned}$$

Therefore, summing over  $n$ , for  $s < 1$ ,

$$\begin{aligned}
J_x(s) &= \sum_{n=0}^{\infty} \mathbf{P}_x \{ \tilde{Z}_n \neq 0 \} s^n \\
&= \sum_{n=0}^{\infty} \gamma_n (1 - (F_n)^x) s^n + \sum_{n=0}^{\infty} \sum_{i=0}^n (\gamma_i - \gamma_{i+1}) \mathbf{P}_x \{ \tilde{Z}_{n-i} \neq 0 \} s^{n+1} \\
&\quad + \sum_{k=1}^{M-2} \sum_{n=0}^{\infty} \sum_{i=0}^n \mathbf{P}_x \{ \tilde{Z}_{n-i} = k \} \gamma_i H_k(F_i) s^{n+1} \\
&= \sum_{n=0}^{\infty} \gamma_n (1 - (F_n)^x) s^n + J_x(s) \sum_{n=0}^{\infty} (\gamma_n - \gamma_{n+1}) s^{n+1} + \sum_{k=1}^{M-2} g_{k,x}(s) \sum_{n=0}^{\infty} \gamma_n H_k(F_n) s^n.
\end{aligned}$$

We conclude the proof noticing that  $\sum_{n=0}^{\infty} (\gamma_n - \gamma_{n+1}) s^{n+1} = (s - 1) \sum_{n=0}^{\infty} \gamma_n s^n + 1$ .  $\square$

We can now give the proof of the proposition.

*Proof of Proposition 3.1.* Recall that the parameter  $\alpha$  is such that  $0 < \alpha \leq 1$ . We first assume  $\alpha < 1$ . Fix  $x \geq 1$  and  $1 \leq k \leq M - 2$ . In view of Lemma 3.3 and with the help of an Abelian/Tauberian theorem (*c.f.* Chap VIII of [5]), we check that

$$(1-s) \sum_{n=0}^{\infty} \gamma_n s^n \underset{s \rightarrow 1^-}{\sim} \frac{c_3 \Gamma(\alpha+1)}{(1-s)^\alpha} \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n H_k(F_n) s^n \underset{s \rightarrow 1^-}{\sim} -\frac{c_3 H'_k(1) \Gamma(\alpha)}{(1-s)^\alpha}.$$

These two equivalences show that  $\tilde{J}_{k,x}(1) \stackrel{\text{def}}{=} \lim_{s \rightarrow 1^-} \tilde{J}_{k,x}(s)$  is finite. More precisely, we get

$$\tilde{J}_{k,x}(1) = -\frac{g_{k,x}(1) H'_k(1)}{\alpha},$$

so that we may write

$$\frac{\tilde{J}_{k,x}(1) - \tilde{J}_{k,x}(s)}{1-s} = \left( \frac{g_{k,x}(1) - g_{k,x}(s)}{1-s} \right) \frac{\tilde{J}_{k,x}(s)}{g_{k,x}(s)} + \frac{g_{k,x}(1) \tilde{B}_k(s)}{(1-s)^2 \sum_{n=0}^{\infty} \gamma_n s^n} \quad (3.4)$$

with the notation

$$\tilde{B}_k(s) \stackrel{\text{def}}{=} \frac{H'_k(1)}{\alpha} (s-1) \sum_{n=0}^{\infty} \gamma_n s^n - \sum_{n=0}^{\infty} \gamma_n H_k(F_n) s^n.$$

The first term on the r.h.s. of (3.4) converges towards  $-g'_k(1) H'_k(1)/\alpha$  as  $s$  tends to 1 (this quantity is finite thanks to Lemma 3.4). Making use of the relation  $\gamma_{n+1} = \delta(F_n) \gamma_n$ , we can also rewrite  $\tilde{B}_k$  in the form

$$\tilde{B}_k(s) = \sum_{n=1}^{\infty} \gamma_{n-1} \left[ \frac{H'_k(1)}{\alpha} (1 - \delta(F_{n-1})) - \delta(F_{n-1}) H_k(F_n) \right] s^n - \frac{H'_k(1)}{\alpha} - H_k(0).$$

With the help of Lemma 3.3, it is easily check that

$$\gamma_{n-1} \left[ \frac{H'_k(1)}{\alpha} (1 - \delta(F_{n-1})) - \delta(F_{n-1}) H_k(F_n) \right] = \mathcal{O} \left( \frac{1}{n^{2-\alpha}} \right).$$

Since  $\alpha < 1$ , we conclude that

$$\tilde{B}_k(1) = \lim_{s \rightarrow 1^-} \tilde{B}_k(s) \quad \text{is finite.} \quad (3.5)$$

We also have

$$(1-s)^2 \sum_{n=0}^{\infty} \gamma_n s^n \underset{s \rightarrow 1^-}{\sim} \frac{c_3 \Gamma(\alpha+1)}{(1-s)^{\alpha-1}}. \quad (3.6)$$

Thus, combining (3.4), (3.5) and (3.6), as  $s \rightarrow 1^-$ ,

$$\frac{\tilde{J}_{k,x}(1) - \tilde{J}_{k,x}(s)}{1-s} = \frac{g_{k,x}(1) \tilde{B}_k(1)}{c_3 \Gamma(\alpha+1)} (1-s)^{\alpha-1} + o((1-s)^{\alpha-1}). \quad (3.7)$$

We can deal with  $\hat{J}_x$  in exactly the same way. We now find  $\hat{J}_x(1) = \frac{x}{\alpha}$  and setting

$$\hat{B}_x(1) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \gamma_{n-1} \left[ \frac{x}{\alpha} (\delta(F_{n-1}) - 1) - \delta(F_{n-1}) (1 - (F_n)^x) \right] + \frac{x}{\alpha} - 1, \quad (3.8)$$

we also find that, as  $s \rightarrow 1^-$ ,

$$\frac{\widehat{J}_x(1) - \widehat{J}_x(s)}{1-s} = \frac{\widehat{B}_x(1)}{c_3\Gamma(\alpha+1)}(1-s)^{\alpha-1} + o((1-s)^{\alpha-1}). \quad (3.9)$$

Putting together (3.7) and (3.9) and using Lemma 3.5, we obtain

$$\frac{J_x(1) - J_x(s)}{1-s} = C_x(1-s)^{\alpha-1} + o((1-s)^{\alpha-1}) \quad (3.10)$$

with

$$C_x \stackrel{\text{def}}{=} \frac{1}{c_3\Gamma(\alpha+1)} \left( \widehat{B}_x(1) + \sum_{k=1}^{M-2} g_{k,x}(1) \widetilde{B}_k(1) \right). \quad (3.11)$$

Since  $x \neq 0$ , we have  $\mathbf{P}_x\{\widetilde{Z}_n \neq 0\} = \mathbf{P}_x\{\sigma > n\}$  and, from the definition of  $J_x$ , we deduce

$$\sum_{n=0}^{\infty} \left( \sum_{k=n+1}^{\infty} \mathbf{P}_x\{\sigma > k\} \right) s^n = \frac{J_x(1) - J_x(s)}{1-s}. \quad (3.12)$$

Combining (3.10) and (3.12), we see that  $C_x \geq 0$ . Moreover, the use of two successive Tauberian theorems yields

$$\mathbf{P}_x\{\sigma > n\} = \frac{C_x \alpha}{\Gamma(1-\alpha)n^{\alpha+1}} + o\left(\frac{1}{n^{\alpha+1}}\right).$$

It remains to prove that  $C_x \neq 0$ . To this end, we first notice that, for  $x, y \geq 0$ , we have  $\mathbf{P}_y\{Z_1 = x\} > 0$  and

$$\mathbf{P}_y\{\sigma > n\} \geq \mathbf{P}_y\{Z_1 = x\} \mathbf{P}_x\{\sigma > n-1\}.$$

Thus,  $C_y \geq \mathbf{P}_y\{Z_1 = x\} C_x$  so it suffices to show that  $C_x$  is not zero for some  $x$ . In view of (a) of Lemma 3.4, the quantity

$$\sum_{k=1}^{M-2} g_{k,x}(1) \widetilde{B}_k(1)$$

is bounded in  $x$ . Looking at the expression of  $C_x$  given in (3.11), it just remains to prove that  $\widehat{B}_x(1)$  can be arbitrarily large. In view of (3.8), we can write

$$\widehat{B}_x(1) = xS(x) + \frac{x}{\alpha} - 1$$

where

$$S(x) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \gamma_{n-1} \left[ \frac{1}{\alpha} (\delta(F_{n-1}) - 1) - \delta(F_{n-1}) \frac{(1 - (F_n)^x)}{x} \right].$$

But for each fixed  $n$ , the function

$$x \rightarrow \delta(F_{n-1}) \frac{(1 - (F_n)^x)}{x}$$

decreases to 0 as  $x$  tends to infinity, so the monotone convergence theorem yields

$$S(x) \underset{x \rightarrow \infty}{\uparrow} \sum_{n=1}^{\infty} \frac{\gamma_{n-1}}{\alpha} (\delta(F_{n-1}) - 1) \sim c_3 \sum_{n=1}^{\infty} \frac{1}{n^{1-\alpha}} = +\infty.$$

Thus,  $\widehat{B}_x(1)$  tends to infinity as  $x$  tends to infinity and the proof of the proposition for  $\alpha < 1$  is complete. The case  $\alpha = 1$  may be treated in a similar fashion (and it is even easier to prove that the constant is not zero). We skip the details.  $\square$

**Remark 3.6.** *The study of the tail distribution of the return time is the key to obtaining conditional limit theorems for the branching process, see for instance [6, 10, 12, 14]. Indeed, following Vatutin's scheme [10] and using Proposition 3.1, it can now be proved that  $Z_n/n$  conditioned on not hitting 0 before time  $n$  converges in law towards an exponential distribution. Precisely, for each  $x = 1, 2, \dots$  and  $r \in \mathbb{R}_+$ ,*

$$\lim_{n \rightarrow \infty} \mathbf{P}_x \left\{ \frac{Z_n}{n} \leq r \mid \sigma > n \right\} = 1 - e^{-r}.$$

*It is to be noted that this result is exactly the same as that obtained for a classical critical Galton-Watson process (i.e. when there is no migration). Although, in our setting, the return time to zero has a finite expectation, which is not the case for the critical Galton-Watson process, the behaviours of both processes conditionally on their non-extinction are still quite similar.*

## 4 Total progeny over an excursion

The aim of this section is to study the distribution of the total progeny of the branching process  $Z$  over an excursion away from 0. We will constantly use the notation

$$\nu \stackrel{\text{def}}{=} \frac{\alpha + 1}{2}.$$

In particular,  $\nu$  ranges through  $(\frac{1}{2}, 1]$ . The main result of this section is the key to the proof of Theorem 1.1 and states as follows.

**Proposition 4.1.** *There exists a constant  $c > 0$  such that*

$$\mathbf{P} \left\{ \sum_{k=0}^{\sigma-1} Z_k > x \right\} \underset{x \rightarrow \infty}{\sim} \begin{cases} c/x^\nu & \text{if } \alpha \in (0, 1) \\ c \log x / x & \text{if } \alpha = 1. \end{cases}$$

Let us first give an informal explanation for this polynomial decay with exponent  $\nu$ . In view of Remark 3.6, we can expect the shape of a large excursion away from zero of the process  $Z$  to be quite similar to that of a Galton-Watson process. Indeed, if  $H$  denotes the height of an excursion of  $Z$  (and  $\sigma$  denotes the length of the excursion), numerical simulations show that, just as in the case of a classical branching process without migration,  $H \approx \sigma$  and the total progeny  $\sum_{k=0}^{\sigma-1} Z_k$  is of the same order as  $H\sigma$ . Since the decay of the tail distribution of  $\sigma$  is polynomial with exponent  $\alpha + 1$ , the

tail distribution of  $\sum_{k=0}^{\sigma-1} Z_k$  should then decrease with exponent  $\frac{\alpha+1}{2}$ . In a way, this proposition tells us that the shape of an excursion is very "squared".

Although there is a vast literature on the subject of branching processes, it seems that there has not been much attention given to the total progeny of the process. Moreover, the classical machinery of generating functions and analytic methods, often used as a rule in the study of branching processes seems, in our setting, inadequate for the study of the total progeny.

The proof of Proposition 4.1 uses a somewhat different approach and is mainly based on a martingale argument. The idea of the proof is fairly simple but, unfortunately, since we are dealing with a discrete time model, a lot of additional technical difficulties appear and the complete argument is quite lengthy. For the sake of clarity, we shall first provide the skeleton of the proof of the proposition, while postponing the proof of the technical estimates to section 5.2.

Let us also note that, although we shall only study the particular branching process associated with the cookie random walk, the method presented here could be used to deal with a more general class of branching processes with migration.

We start with an easy lemma stating that  $\mathbf{P}\{\sum_{k=0}^{\sigma-1} Z_k > x\}$  cannot decrease much faster than  $\frac{1}{x^\nu}$ .

**Lemma 4.2.** *For any  $\beta > \nu$ , we have*

$$\mathbf{E} \left[ \left( \sum_{k=0}^{\sigma-1} Z_k \right)^\beta \right] = \infty.$$

*Proof.* When  $\alpha = \nu = 1$ , the result is a direct consequence of Corollary 2.3 of section 2. We now assume  $\alpha < 1$ . Hölder's inequality gives

$$\sum_{n=0}^{\sigma-1} Z_n^\alpha \leq \sigma^{1-\alpha} \left( \sum_{n=0}^{\sigma-1} Z_n \right)^\alpha.$$

Taking the expectation and applying again Hölder's inequality, we obtain, for  $\varepsilon > 0$  small enough

$$\mathbf{E} \left[ \sum_{n=0}^{\sigma-1} Z_n^\alpha \right] \leq \mathbf{E}[\sigma^{1+\alpha-\varepsilon}]^{\frac{1}{p}} \mathbf{E} \left[ \left( \sum_{n=0}^{\sigma-1} Z_n \right)^{\alpha q} \right]^{\frac{1}{q}},$$

with  $p = \frac{1+\alpha-\varepsilon}{1-\alpha}$  and  $\alpha q = \frac{1+\alpha-\varepsilon}{2-\varepsilon/\alpha}$ . Moreover, Corollary 2.3 states that  $\mathbf{E}[\sum_{n=0}^{\sigma-1} Z_n^\alpha] = \infty$  and thanks to Corollary 3.2,  $\mathbf{E}[\sigma^{1+\alpha-\varepsilon}] < \infty$ . Therefore,

$$\mathbf{E} \left[ \left( \sum_{n=0}^{\sigma-1} Z_n \right)^{\alpha q} \right] = \mathbf{E} \left[ \left( \sum_{n=0}^{\sigma-1} Z_n \right)^{\nu+\varepsilon'} \right] = \infty.$$

This result is valid for any  $\varepsilon'$  small enough and completes the proof of the lemma.  $\square$

*Proof of Proposition 4.1.* Let us first note that, in view of an Abelian/Tauberian theorem, Proposition 4.1 is equivalent to

$$\mathbf{E} \left[ 1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k} \right] \underset{\lambda \rightarrow 0^+}{\sim} \begin{cases} C\lambda^\nu & \text{if } \alpha \in (0, 1), \\ C\lambda \log \lambda & \text{if } \alpha = 1, \end{cases}$$

where  $C$  is a positive constant. We now construct a martingale in the following way. Let  $K_\nu$  denote the modified Bessel function of second kind with parameter  $\nu$ . For  $\lambda > 0$ , we define

$$\phi_\lambda(x) \stackrel{\text{def}}{=} (\sqrt{\lambda}x)^\nu K_\nu(\sqrt{\lambda}x), \quad \text{for } x > 0. \quad (4.1)$$

We shall give some important properties of  $\phi_\lambda$  in section 5.1. For the time being, we simply recall that  $\phi_\lambda$  is an analytic, positive, decreasing function on  $(0, \infty)$  such that  $\phi_\lambda$  and  $\phi'_\lambda$  are continuous at 0 with

$$\phi_\lambda(0) = 2^{\nu-1}\Gamma(\nu) \quad \text{and} \quad \phi'_\lambda(0) = 0. \quad (4.2)$$

Our main interest in  $\phi_\lambda$  is that it satisfies the following differential equation, for  $x > 0$ :

$$-\lambda x \phi_\lambda(x) - \alpha \phi'_\lambda(x) + x \phi''_\lambda(x) = 0. \quad (4.3)$$

Now let  $(\mathcal{F}_n, n \geq 0)$  denote the natural filtration of the branching process  $Z$  *i.e.*  $\mathcal{F}_n \stackrel{\text{def}}{=} \sigma(Z_k, 0 \leq k \leq n)$  and define, for  $n \geq 0$  and  $\lambda > 0$ ,

$$W_n \stackrel{\text{def}}{=} \phi_\lambda(Z_n) e^{-\lambda \sum_{k=0}^{n-1} Z_k}. \quad (4.4)$$

Setting

$$\mu(n) \stackrel{\text{def}}{=} \mathbf{E}[W_n - W_{n+1} \mid \mathcal{F}_n], \quad (4.5)$$

it is clear that the process

$$Y_n \stackrel{\text{def}}{=} W_n + \sum_{k=0}^{n-1} \mu(k)$$

is an  $\mathcal{F}$ -martingale. Furthermore, this martingale has bounded increments since

$$|Y_{n+1} - Y_n| \leq |W_{n+1} - W_n| + |\mu(n)| \leq 4\|\phi_\lambda\|_\infty.$$

Therefore, the use of the optional sampling theorem is legitimate with any stopping time with finite mean. In particular, applying the optional sampling theorem with the first return time to 0, we get

$$\phi_\lambda(0) \mathbf{E}[e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}] = \phi_\lambda(0) - \mathbf{E}\left[\sum_{k=0}^{\sigma-1} \mu(k)\right],$$

which we may be rewritten, using that  $\phi_\lambda(0) = 2^{\nu-1}\Gamma(\nu)$ ,

$$\mathbf{E}[1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}] = \frac{1}{2^{\nu-1}\Gamma(\nu)} \mathbf{E}\left[\sum_{k=0}^{\sigma-1} \mu(k)\right]. \quad (4.6)$$

The proof of Proposition 4.1 now relies on a careful study of the expectation of  $\sum_{k=0}^{\sigma-1} \mu(k)$ . To this end, we shall decompose  $\mu$  into several terms using a Taylor expansion of  $\phi_\lambda$ . We first need the following lemma:

**Lemma 4.3.**

(a) *There exists a function  $f_1$  with  $f_1(x) = 0$  for all  $x \geq M - 1$  such that*

$$\mathbf{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] = -\alpha + f_1(Z_n).$$

(b) *There exists a function  $f_2$  with  $f_2(x) = f_2(M - 1)$  for all  $x \geq M - 1$  such that*

$$\mathbf{E}[(Z_{n+1} - Z_n)^2 \mid \mathcal{F}_n] = 2Z_n + 2f_2(Z_n).$$

(c) *For  $p \in \mathbb{N}^*$ , there exists a constant  $D_p$  such that*

$$\mathbf{E}[|Z_{n+1} - Z_n|^p \mid \mathcal{F}_n] \leq D_p(Z_n^{p/2} + \mathbb{1}_{\{Z_n=0\}}).$$

*Proof.* Assertion (a) is just a rewriting of equation (2.4). Recall the notations introduced in section 2. Recall in particular that  $\mathbf{E}[A_{M-1}] = M - 1 - \alpha$ . Thus, for  $j \geq M - 1$ , we have

$$\begin{aligned} \mathbf{E}[(Z_{n+1} - Z_n)^2 \mid Z_n = j] &= \mathbf{E}[(A_{M-1} + \xi_1 + \dots + \xi_{j-M+1} - j)^2] \\ &= \mathbf{E}\left[\left(\alpha + (A_{M-1} - \mathbf{E}[A_{M-1}]) + \sum_{k=1}^{j-M+1} (\xi_k - \mathbf{E}[\xi_k])\right)^2\right] \\ &= \alpha^2 + \mathbf{Var}(A_{M-1}) + (j - M + 1)\mathbf{Var}(\xi_1) \\ &= 2Z_n + \alpha^2 + \mathbf{Var}(A_{M-1}) - 2(M - 1). \end{aligned}$$

This proves (b). When  $p$  is an even integer, we have  $\mathbf{E}[|Z_{n+1} - Z_n|^p \mid \mathcal{F}_n] = \mathbf{E}[(Z_{n+1} - Z_n)^p \mid \mathcal{F}_n]$  and assertion (c) can be proved by developing  $(Z_{n+1} - Z_n)^p$  in the same manner as for (b). Finally, when  $p$  is an odd integer, Hölder's inequality gives

$$\mathbf{E}[|Z_{n+1} - Z_n|^p \mid Z_n = j > 0] \leq \mathbf{E}[|Z_{n+1} - Z_n|^{p+1} \mid Z_n = j > 0]^{\frac{p}{p+1}} \leq D_{\frac{p}{p+1}}^{\frac{p}{p+1}} Z_n^{\frac{p}{2}}.$$

□

*Continuation of the proof of Proposition 4.1.* For  $n \in [1, \sigma - 2]$ , the random variables  $Z_n$  and  $Z_{n+1}$  are both non zero and, since  $\phi_\lambda$  is infinitely differentiable on  $(0, \infty)$ , a Taylor expansion yields

$$\phi_\lambda(Z_{n+1}) = \phi_\lambda(Z_n) + \phi'_\lambda(Z_n)(Z_{n+1} - Z_n) + \frac{1}{2}\phi''_\lambda(Z_n)(Z_{n+1} - Z_n)^2 + \theta_n, \quad (4.7)$$

where  $\theta_n$  is given by Taylor's integral remainder formula

$$\theta_n \stackrel{\text{def}}{=} (Z_{n+1} - Z_n)^2 \int_0^1 (1-t)(\phi''_\lambda(Z_n + t(Z_{n+1} - Z_n)) - \phi''_\lambda(Z_n))dt. \quad (4.8)$$

When  $n = \sigma - 1$ , this result is *a priori* incorrect because then  $Z_{n+1} = 0$ . However, according to (4.2) and (4.3), the functions  $\phi_\lambda(t)$ ,  $\phi'_\lambda(t)$  and  $t\phi''_\lambda(t)$  have finite limits as  $t$  tends to  $0^+$ , thus equation (4.7) still holds when  $n = \sigma - 1$ . Therefore, for  $n \in [1, \sigma - 1]$ ,

$$\begin{aligned} \mathbf{E}[e^{\lambda Z_n} \phi_\lambda(Z_n) - \phi_\lambda(Z_{n+1}) \mid \mathcal{F}_n] &= \\ (e^{\lambda Z_n} - 1)\phi_\lambda(Z_n) - \phi'_\lambda(Z_n)\mathbf{E}[Z_{n+1} - Z_n \mid \mathcal{F}_n] - \frac{1}{2}\phi''_\lambda(Z_n)\mathbf{E}[(Z_{n+1} - Z_n)^2 \mid \mathcal{F}_n] - \mathbf{E}[\theta_n \mid \mathcal{F}_n]. \end{aligned}$$



In view of (a) and (b) of Lemma 4.3 and recalling the differential equation (4.3) satisfied by  $\phi_\lambda$ , the r.h.s. of the previous equality may be rewritten

$$(e^{\lambda Z_n} - 1 - \lambda Z_n)\phi_\lambda(Z_n) - \phi'_\lambda(Z_n)f_1(Z_n) - \phi''_\lambda(Z_n)f_2(Z_n) - \mathbf{E}[\theta_n \mid \mathcal{F}_n].$$

On the other hand, in view of (4.4) and (4.5), we have

$$\mu(n) = e^{-\lambda \sum_{k=0}^n Z_k} \mathbf{E}[e^{\lambda Z_n} \phi_\lambda(Z_n) - \phi_\lambda(Z_{n+1}) \mid \mathcal{F}_n]. \quad (4.9)$$

Thus, for each  $n \in [1, \sigma - 1]$ , we may decompose  $\mu(n)$  in the form

$$\mu(n) = \mu_1(n) + \mu_2(n) + \mu_3(n) + \mu_4(n), \quad (4.10)$$

where

$$\begin{aligned} \mu_1(n) &\stackrel{\text{def}}{=} e^{-\lambda \sum_{k=0}^n Z_k} (e^{\lambda Z_n} - 1 - \lambda Z_n) \phi_\lambda(Z_n) \\ \mu_2(n) &\stackrel{\text{def}}{=} -e^{-\lambda \sum_{k=0}^n Z_k} \phi'_\lambda(Z_n) f_1(Z_n) \\ \mu_3(n) &\stackrel{\text{def}}{=} -e^{-\lambda \sum_{k=0}^n Z_k} \phi''_\lambda(Z_n) f_2(Z_n) \\ \mu_4(n) &\stackrel{\text{def}}{=} -e^{-\lambda \sum_{k=0}^n Z_k} \mathbf{E}[\theta_n \mid \mathcal{F}_n]. \end{aligned}$$

In particular, we can rewrite (4.6) in the form (we have to treat  $\mu(0)$  separately since (4.8) does not hold for  $n = 0$ )

$$\mathbf{E}[1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}] = \frac{1}{2^{\nu-1} \Gamma(\nu)} \left( \mathbf{E}[\mu(0)] + \sum_{i=1}^4 \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_i(n) \right] \right). \quad (4.11)$$

We now state the main estimates:

**Lemma 4.4.** *There exist  $\varepsilon > 0$  and eight finite constants  $(C_i, C'_i, i = 0, 2, 3, 4)$  such that, as  $\lambda$  tends to  $0^+$ ,*

$$\begin{aligned} \text{(a)} \quad \mathbf{E}[\mu(0)] &= \begin{cases} C_0 \lambda^\nu + \mathcal{O}(\lambda) & \text{if } \alpha \in (0, 1) \\ C_0 \lambda \log \lambda + C'_0 \lambda + o(\lambda) & \text{if } \alpha = 1, \end{cases} \\ \text{(b)} \quad \mathbf{E}[\sum_{n=1}^{\sigma-1} \mu_1(n)] &= o(\lambda) \quad \text{for } \alpha \in (0, 1], \\ \text{(c)} \quad \mathbf{E}[\sum_{n=1}^{\sigma-1} \mu_2(n)] &= \begin{cases} C_2 \lambda^\nu + o(\lambda^{\nu+\varepsilon}) & \text{if } \alpha \in (0, 1) \\ C_2 \lambda \log \lambda + C'_2 \lambda + o(\lambda) & \text{if } \alpha = 1, \end{cases} \\ \text{(d)} \quad \mathbf{E}[\sum_{n=1}^{\sigma-1} \mu_3(n)] &= \begin{cases} C_3 \lambda^\nu + o(\lambda^{\nu+\varepsilon}) & \text{if } \alpha \in (0, 1) \\ C_3 \lambda \log \lambda + C'_3 \lambda + o(\lambda) & \text{if } \alpha = 1, \end{cases} \\ \text{(e)} \quad \mathbf{E}[\sum_{n=1}^{\sigma-1} \mu_4(n)] &= \begin{cases} C_4 \lambda^\nu + o(\lambda^{\nu+\varepsilon}) & \text{if } \alpha \in (0, 1) \\ C'_4 \lambda + o(\lambda) & \text{if } \alpha = 1. \end{cases} \end{aligned}$$

Let us for the time being postpone the long and technical proof of these estimates until section 5.2 and complete the proof of Proposition 4.1. In view of (4.11), using the previous lemma, we deduce that there exist some constants  $C, C'$  such that

$$\mathbf{E} \left[ 1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k} \right] = \begin{cases} C \lambda^\nu + o(\lambda^{\nu+\varepsilon}) & \text{if } \alpha \in (0, 1), \\ C \lambda \log \lambda + C' \lambda + o(\lambda) & \text{if } \alpha = 1. \end{cases} \quad (4.12)$$

with

$$C \stackrel{\text{def}}{=} \begin{cases} 2^{1-\nu}\Gamma(\nu)^{-1}(C_0 + C_2 + C_3 + C_4) & \text{when } \alpha < 1, \\ 2^{1-\nu}\Gamma(\nu)^{-1}(C_0 + C_2 + C_3) & \text{when } \alpha = 1. \end{cases}$$

It simply remains to check that the constant  $C$  is not zero. Indeed, suppose that  $C = 0$ . We first assume  $\alpha = 1$ . Then, from (4.12),

$$\mathbf{E} \left[ 1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k} \right] = C' \lambda + o(\lambda)$$

which implies  $\mathbf{E}[\sum_{k=0}^{\sigma-1} Z_k] < \infty$  and contradicts Corollary 2.3. Similarly, when  $\alpha \in (0, 1)$  and  $C = 0$ , we get from (4.12),

$$\mathbf{E} \left[ 1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k} \right] = o(\lambda^{\nu+\varepsilon}).$$

This implies, for any  $0 < \varepsilon' < \varepsilon$ , that

$$\mathbf{E} \left[ \left( \sum_{n=0}^{\sigma-1} Z_n \right)^{\nu+\varepsilon'} \right] < \infty$$

which contradicts Lemma 4.2. Therefore,  $C$  cannot be zero and the proposition is proved.  $\square$

## 5 Technical estimates

### 5.1 Some properties of modified Bessel functions

We now recall some properties of modified Bessel functions. All the results cited here may be found in [1] (section 9.6) or [8] (section 5.7). For  $\eta \in \mathbb{R}$ , the modified Bessel function of the first kind  $I_\eta$  is defined by

$$I_\eta(x) \stackrel{\text{def}}{=} \left( \frac{x}{2} \right)^\eta \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{\Gamma(k+1)\Gamma(k+1+\eta)}$$

and the modified Bessel function of the second kind  $K_\eta$  is given by the formula

$$K_\eta(x) \stackrel{\text{def}}{=} \begin{cases} \frac{\pi}{2} \frac{I_{-\eta}(x) - I_\eta(x)}{\sin \pi \eta} & \text{for } \eta \in \mathbb{R} - \mathbb{Z}, \\ \lim_{\eta' \rightarrow \eta} K_{\eta'}(x) & \text{for } \eta \in \mathbb{Z}. \end{cases}$$

We are particularly interested in

$$F_\eta(x) \stackrel{\text{def}}{=} x^\eta K_\eta(x) \quad \text{for } x > 0.$$

Thus, the function  $\phi_\lambda$  defined in (4.1) may be expressed in the form

$$\phi_\lambda(x) = F_\nu(\sqrt{\lambda}x). \tag{5.1}$$

**Fact 5.1.** *For  $\eta \geq 0$ , the function  $F_\eta$  is analytic, positive and strictly decreasing on  $(0, \infty)$ . Moreover*

1. Behaviour at 0:

- (a) If  $\eta > 0$ , the function  $F_\eta$  is defined by continuity at 0 with  $F_\eta(0) = 2^{\eta-1}\Gamma(\eta)$ .
- (b) If  $\eta = 0$ , then  $F_0(x) = -\log x + \log 2 - \gamma + o(1)$  as  $x \rightarrow 0^+$  where  $\gamma$  denotes Euler's constant.

2. Behaviour at infinity:

$$F_\eta(x) \underset{x \rightarrow \infty}{\sim} \sqrt{\frac{\pi}{2x}} e^{-x}.$$

In particular, for every  $\eta > 0$ , there exists  $c_\eta \in \mathbb{R}$  such that, for all  $x \geq 0$ ,

$$F_\eta(x) \leq c_\eta e^{-x}. \quad (5.2)$$

3. Formula for the derivative:

$$F'_\eta(x) = -x^{2\eta-1} F_{1-\eta}(x). \quad (5.3)$$

In particular,  $F_\eta$  solves the differential equation

$$xF''_\eta(x) - (2\eta - 1)F'_\eta(x) - xF_\eta(x) = 0.$$

Concerning the function  $\phi_\lambda$ , in view of (5.1), we deduce

**Fact 5.2.** *For each  $\lambda > 0$ , the function  $\phi_\lambda$  is analytic, positive and strictly decreasing on  $(0, \infty)$ . Moreover*

- (a)  $\phi_\lambda$  is continuous and differentiable at 0 with  $\phi_\lambda(0) = 2^{\nu-1}\Gamma(\nu)$  and  $\phi'_\lambda(0) = 0$ .
- (b) For  $x > 0$ , we have

$$\begin{aligned} \phi'_\lambda(x) &= -\lambda^\nu x^\alpha F_{1-\nu}(\sqrt{\lambda}x), \\ \phi''_\lambda(x) &= \lambda F_\nu(\sqrt{\lambda}x) - \alpha \lambda^\nu x^{\alpha-1} F_{1-\nu}(\sqrt{\lambda}x). \end{aligned}$$

In particular,  $\phi_\lambda$  solves the differential equation

$$-\lambda x \phi_\lambda(x) - \alpha \phi'_\lambda(x) + x \phi''_\lambda(x) = 0.$$

## 5.2 Proof of Lemma 4.4

The proof of Lemma 4.4 is long and tedious but requires only elementary methods. We shall treat, in separate subsections the assertions (a) - (e) when  $\alpha < 1$ . We explain, in a last subsection, how to deal with the case  $\alpha = 1$ .

We will use the following result extensively throughout the proof of Lemma 4.4.

**Lemma 5.3.** *There exists  $\varepsilon > 0$  such that*

$$\mathbf{E} \left[ \sigma(1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}) \right] = o(\lambda^\varepsilon) \quad \text{as } \lambda \rightarrow 0^+.$$

*Proof.* Let  $\beta < \alpha \leq 1$ , the function  $x \rightarrow x^\beta$  is concave, thus

$$\mathbf{E} \left[ \left( \sum_{k=0}^{\sigma-1} Z_k \right)^\beta \right] \leq \mathbf{E} \left[ \sum_{k=0}^{\sigma-1} Z_k^\beta \right] \stackrel{\text{def}}{=} c_1 < \infty,$$

where we used Corollary 2.3 to conclude on the finiteness of  $c_1$ . From Markov's inequality, we deduce that  $\mathbf{P} \left\{ \sum_{k=0}^{\sigma-1} Z_k > x \right\} \leq \frac{c_1}{x^\beta}$  for all  $x \geq 0$ . Therefore,

$$\mathbf{E} \left[ 1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k} \right] \leq (1 - e^{-\lambda x}) + \mathbf{P} \left\{ \sum_{k=0}^{\sigma-1} Z_k > x \right\} \leq \lambda x + \frac{c_1}{x^\beta}.$$

Choosing  $x = \lambda^{-\frac{1}{\beta+1}}$  and setting  $\beta' \stackrel{\text{def}}{=} \frac{\beta}{\beta+1}$ , we deduce

$$\mathbf{E} \left[ 1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k} \right] \leq (1 + c_1) \lambda^{\beta'}.$$

According to Corollary 3.2, for  $\delta < \alpha$ , we have  $\mathbf{E}[\sigma^{1+\delta}] < \infty$ , so Hölder's inequality gives

$$\begin{aligned} \mathbf{E} \left[ \sigma (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}) \right] &\leq \mathbf{E}[\sigma^{1+\delta}]^{\frac{1}{1+\delta}} \mathbf{E} \left[ (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k})^{\frac{1+\delta}{\delta}} \right]^{\frac{\delta}{1+\delta}} \\ &\leq \mathbf{E}[\sigma^{1+\delta}]^{\frac{1}{1+\delta}} \mathbf{E} \left[ 1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k} \right]^{\frac{\delta}{1+\delta}} \leq c_2 \lambda^{\frac{\beta' \delta}{1+\delta}}, \end{aligned}$$

which completes the proof of the lemma.  $\square$

### 5.2.1 Proof of (a) of Lemma 4.4 when $\alpha < 1$

Using the expression of  $\mu(0)$  given by (4.9) and the relation (5.3) between of  $F'_\nu$  and  $F_{1-\nu}$ , we have

$$\mathbf{E}[\mu(0)] = \mathbf{E}[F_\nu(0) - F_\nu(\sqrt{\lambda} Z_1)] = -\mathbf{E} \left[ \int_0^{\sqrt{\lambda} Z_1} F'_\nu(x) dx \right] = \lambda^\nu \mathbf{E} \left[ \int_0^{Z_1} y^\alpha F_{1-\nu}(\sqrt{\lambda} y) dy \right].$$

Thus, using the dominated convergence theorem,

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda^\nu} E[\mu(0)] = \mathbf{E} \left[ \int_0^{Z_1} y^\alpha F_{1-\nu}(0) dy \right] = \frac{F_{1-\nu}(0)}{1+\alpha} \mathbf{E}[Z_1^{1+\alpha}] \stackrel{\text{def}}{=} C_0 < \infty.$$

Furthermore, using again (5.3), we get

$$\begin{aligned} \left| \frac{1}{\lambda^\nu} E[\mu(0)] - C_0 \right| &= \mathbf{E} \left[ \int_0^{Z_1} y^\alpha \left( F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda} y) \right) dy \right] \\ &= \mathbf{E} \left[ \int_0^{Z_1} y^\alpha \int_0^{\sqrt{\lambda} y} x^{-\alpha} F_\nu(x) dx dy \right] \\ &\leq \frac{\|F_\nu\|_\infty}{1-\alpha} \lambda^{\frac{1-\alpha}{2}} \mathbf{E} \left[ \int_0^{Z_1} y dy \right] = \frac{\|F_\nu\|_\infty \mathbf{E}[Z_1^2]}{2(1-\alpha)} \lambda^{\frac{1-\alpha}{2}}. \end{aligned}$$

Therefore, we obtain

$$E[\mu(0)] = C_0 \lambda^\nu + \mathcal{O}(\lambda)$$

which proves (a) of Lemma 4.4.

### 5.2.2 Proof of (b) of Lemma 4.4 when $\alpha < 1$

Recall that

$$\mu_1(n) = e^{-\lambda \sum_{k=0}^n Z_k} (e^{\lambda Z_n} - 1 - \lambda Z_n) \phi_\lambda(Z_n) = e^{-\lambda \sum_{k=0}^n Z_k} (e^{\lambda Z_n} - 1 - \lambda Z_n) F_\nu(\sqrt{\lambda} Z_n).$$

Thus,  $\mu_1(n)$  is almost surely positive and

$$\mu_1(n) \leq (1 - e^{-\lambda Z_n} - \lambda Z_n e^{-\lambda Z_n}) F_\nu(\sqrt{\lambda} Z_n).$$

Moreover, for any  $y > 0$ , we have  $1 - e^{-y} - ye^{-y} \leq \min(1, y^2)$ , thus

$$\begin{aligned} \mu_1(n) &\leq (1 - e^{-\lambda Z_n} - \lambda Z_n e^{-\lambda Z_n}) F_\nu(\sqrt{\lambda} Z_n) \left( \mathbb{1}_{\{Z_n > \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} + \mathbb{1}_{\{Z_n \leq \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} \right) \\ &\leq F_\nu(\sqrt{\lambda} Z_n) \mathbb{1}_{\{Z_n > \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} + \|F_\nu\|_\infty \lambda^2 Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} \\ &\leq F_\nu(-2 \log \lambda) + \|F_\nu\|_\infty \lambda^2 Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2 \log \lambda}{\sqrt{\lambda}}\}}, \end{aligned}$$

where we used the fact that  $F_\nu$  is decreasing for the last inequality. In view of (5.2), we also have  $F_\nu(-2 \log \lambda) \leq c_\nu \lambda^2$  and therefore

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_1(n) \right] \leq \lambda^2 c_\nu \mathbf{E}[\sigma] + \lambda^2 \|F_\nu\|_\infty \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} \right]. \quad (5.4)$$

On the one hand, according to (2.5), we have

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} \right] = \mathbf{E} \left[ Z_\infty^2 \mathbb{1}_{\{Z_\infty \leq \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} \right] \mathbf{E}[\sigma]. \quad (5.5)$$

On the other hand, Proposition 2.2 states that  $\mathbf{P}(Z_\infty \geq x) \sim \frac{C}{x^\alpha}$  as  $x$  tends to infinity, thus

$$\mathbf{E} [Z_\infty^2 \mathbb{1}_{\{Z_\infty \leq x\}}] \underset{x \rightarrow \infty}{\sim} 2 \sum_{k=1}^x k \mathbf{P}(Z_\infty \geq k) \underset{x \rightarrow \infty}{\sim} \frac{2C}{2-\alpha} x^{2-\alpha}.$$

This estimate and (5.5) yield

$$\lambda^2 \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} \right] \underset{\lambda \rightarrow 0+}{\sim} c_3 \lambda^{1+\frac{\alpha}{2}} |\log \lambda|^{2-\alpha}. \quad (5.6)$$

Combining (5.4) and (5.6), we finally obtain

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_1(n) \right] = o(\lambda),$$

which proves (b) of Lemma 4.4.

### 5.2.3 Proof of (c) of Lemma 4.4 when $\alpha < 1$

Recall that

$$\mu_2(n) = -e^{-\lambda \sum_{k=0}^n Z_k} \phi'_\lambda(Z_n) f_1(Z_n) = \lambda^\nu Z_n^\alpha F_{1-\nu}(\sqrt{\lambda} Z_n) f_1(Z_n) e^{-\lambda \sum_{k=0}^n Z_k}.$$

Since  $f_1(x) = 0$  for  $x \geq M - 1$  (c.f. Lemma 4.3), the quantity  $|\mu_2(n)|/\lambda^\nu$  is smaller than  $M^\alpha \|f_1\|_\infty \|F_{1-\nu}\|_\infty$ . Thus, using the dominated convergence theorem, we get

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda^\nu} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_2(n) \right] = \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^\alpha F_{1-\nu}(0) f_1(Z_n) \right] \stackrel{\text{def}}{=} C_2 \in \mathbb{R}.$$

It remains to prove that, for  $\varepsilon > 0$  small enough, as  $\lambda \rightarrow 0^+$

$$\left| \frac{1}{\lambda^\nu} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_2(n) \right] - C_2 \right| = o(\lambda^\varepsilon). \quad (5.7)$$

We can rewrite the l.h.s. of (5.7) in the form

$$\begin{aligned} & \left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^\alpha f_1(Z_n) (F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda} Z_n)) \right] \right. \\ & \quad \left. + \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^\alpha f_1(Z_n) F_{1-\nu}(\sqrt{\lambda} Z_n) (1 - e^{-\lambda \sum_{k=0}^n Z_k}) \right] \right|. \quad (5.8) \end{aligned}$$

On the one hand, the first term is bounded by

$$\begin{aligned} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^\alpha |f_1(Z_n)| (F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda} Z_n)) \right] & \leq M^\alpha \|f_1\|_\infty \mathbf{E}[\sigma] \int_0^{\sqrt{\lambda} M} |F'_{1-\nu}(x)| dx \\ & \leq M^\alpha \|f_1\|_\infty \mathbf{E}[\sigma] \|F_\nu\|_\infty \int_0^{\sqrt{\lambda} M} x^{1-2\nu} dx \\ & \leq c_4 \lambda^{1-\nu}, \end{aligned}$$

where we used formula (5.3) for the expression of  $F'_{1-\nu}$  for the second inequality. On the other hand the second term of (5.8) is bounded by

$$\begin{aligned} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^\alpha |f_1(Z_n)| F_{1-\nu}(\sqrt{\lambda} Z_n) (1 - e^{-\lambda \sum_{k=0}^n Z_k}) \right] & \leq M^\alpha \|f_1\|_\infty \|F_{1-\nu}\|_\infty \mathbf{E}[\sigma (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k})] \\ & \leq c_5 \lambda^\varepsilon \end{aligned}$$

where we used Lemma 5.3 for the last inequality. Putting the pieces together, we conclude that (5.7) holds for  $\varepsilon > 0$  small enough.

### 5.2.4 Proof of (d) of Lemma 4.4 when $\alpha < 1$

Recall that

$$\begin{aligned}\mu_3(n) &= -e^{-\lambda \sum_{k=0}^n Z_k} \phi''_\lambda(Z_n) f_2(Z_n) \\ &= -e^{-\lambda \sum_{k=0}^n Z_k} f_2(Z_n) \left( \lambda F_\nu(\sqrt{\lambda} Z_n) + \alpha \lambda^\nu Z_n^{\alpha-1} F_{1-\nu}(\sqrt{\lambda} Z_n) \right).\end{aligned}$$

Note that, since  $\alpha \leq 1$ , we have  $Z_n^{\alpha-1} \leq 1$  when  $Z_n \neq 0$ . The quantities  $f_2(Z_n)$ ,  $F_\nu(\sqrt{\lambda} Z_n)$  and  $F_{1-\nu}(\sqrt{\lambda} Z_n)$  are also bounded, so we check, using the dominated convergence theorem, that

$$\lim_{\lambda \rightarrow 0} \frac{1}{\lambda^\nu} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_3(n) \right] = -\alpha \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} F_{1-\nu}(0) f_2(Z_n) \right] \stackrel{\text{def}}{=} C_3 \in \mathbb{R}.$$

Furthermore we have

$$\begin{aligned}\frac{1}{\lambda^\nu} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_3(n) \right] - C_3 &= -\lambda^{1-\nu} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} f_2(Z_n) F_\nu(\sqrt{\lambda} Z_n) \right] \\ &\quad + \alpha \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} f_2(Z_n) \left( F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda} Z_n) \right) \right] \\ &\quad + \alpha \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} f_2(Z_n) F_{1-\nu}(\sqrt{\lambda} Z_n) \left( 1 - e^{-\lambda \sum_{k=0}^n Z_k} \right) \right].\end{aligned}\tag{5.9}$$

The first term is clearly bounded by  $c_6 \lambda^{1-\nu}$ . We turn our attention to the second term. In view of (5.3), we have

$$F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda} Z_n) = \int_0^{\sqrt{\lambda} Z_n} x^{1-2\nu} F_\nu(x) dx \leq \frac{\|F_\nu\|_\infty}{2-2\nu} \lambda^{1-\nu} Z_n^{2-2\nu} = \frac{\|F_\nu\|_\infty}{1-\alpha} \lambda^{1-\nu} Z_n^{1-\alpha},$$

where we used  $2-2\nu = 1-\alpha$  for the last equality. Therefore,

$$\begin{aligned}\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} f_2(Z_n) (F_{1-\nu}(0) - F_{1-\nu}(\sqrt{\lambda} Z_n)) \right] \right| &\leq \frac{\|F_\nu\|_\infty \|f_2\|_\infty}{1-\alpha} \lambda^{1-\nu} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} 1 \right] \\ &\leq \frac{\|F_\nu\|_\infty \|f_2\|_\infty \mathbf{E}[\sigma]}{1-\alpha} \lambda^{1-\nu}.\end{aligned}$$

As for the third term of (5.9), with the help of Lemma 5.3, we find

$$\begin{aligned}\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\alpha-1} f_2(Z_n) F_{1-\nu}(\sqrt{\lambda} Z_n) (1 - e^{-\lambda \sum_{k=0}^n Z_k}) \right] \right| &\leq \|f_2\|_\infty \|F_{1-\nu}\|_\infty \mathbf{E} \left[ \sigma (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}) \right] \\ &\leq c_7 \lambda^\varepsilon.\end{aligned}$$

Putting the pieces together, we conclude that

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_3(n) \right] = C_3 \lambda^\nu + o(\lambda^{\nu+\varepsilon}).$$

### 5.2.5 Proof of (e) of Lemma 4.4 when $\alpha < 1$

Recall that

$$\mu_4(n) = -e^{-\lambda \sum_{k=0}^n Z_k} \mathbf{E}[\theta_n \mid \mathcal{F}_n]. \quad (5.10)$$

This term is clearly the most difficult to deal with. We first need the next lemma stating that  $Z_{n+1}$  cannot be too "far" from  $Z_n$ .

**Lemma 5.4.** *There exist two constants  $K_1, K_2 > 0$  such that for all  $n \geq 0$ ,*

$$(a) \quad \mathbf{P}(Z_{n+1} \leq \tfrac{1}{2}Z_n \mid \mathcal{F}_n) \leq K_1 e^{-K_2 Z_n},$$

$$(b) \quad \mathbf{P}(Z_{n+1} \geq 2Z_n \mid \mathcal{F}_n) \leq K_1 e^{-K_2 Z_n}.$$

*Proof.* This lemma follows from large deviation estimates. Indeed, with the notation of section 2, in view of Cramer's theorem, we have, for any  $j \geq M - 1$ ,

$$\begin{aligned} \mathbf{P}\left\{Z_{n+1} \leq \tfrac{1}{2}Z_n \mid Z_n = j\right\} &= \mathbf{P}\left\{A_{M-1} + \xi_1 + \dots + \xi_{j-M+1} \leq \tfrac{j}{2}\right\} \\ &\leq \mathbf{P}\left\{\xi_1 + \dots + \xi_{j-M+1} \leq \tfrac{j}{2}\right\} \leq K_1 e^{-K_2 j}, \end{aligned}$$

where we used the fact that  $(\xi_i)$  is a sequence of i.i.d geometric random variables with mean 1. Similarly, recalling that  $A_{M-1}$  admits exponential moments of order  $\beta < 2$ , we also deduce, for  $j \geq M - 1$ , with possibly extended values of  $K_1$  and  $K_2$ , that

$$\begin{aligned} \mathbf{P}\left\{Z_{n+1} \geq 2Z_n \mid Z_n = j\right\} &= \mathbf{P}\left\{A_{M-1} + \xi_1 + \dots + \xi_{j-M+1} \geq 2j\right\} \\ &\leq \mathbf{P}\left\{A_{M-1} \geq \tfrac{j}{2}\right\} + \mathbf{P}\left\{\xi_1 + \dots + \xi_{j-M+1} \geq \tfrac{3j}{2}\right\} \leq K_1 e^{-K_2 j}. \end{aligned}$$

□

Throughout this section, we use the notation, for  $t \in [0, 1]$  and  $n \in \mathbb{N}$ ,

$$V_{n,t} \stackrel{\text{def}}{=} Z_n + t(Z_{n+1} - Z_n).$$

In particular  $V_{n,t} \in [Z_n, Z_{n+1}]$  (with the convention that for  $a > b$ ,  $[a, b]$  means  $[b, a]$ ). With this notation, we can rewrite the expression of  $\theta_n$  given in (4.8) in the form

$$\theta_n = (Z_{n+1} - Z_n)^2 \int_0^1 (1-t) \left( \phi''_\lambda(V_{n,t}) - \phi''_\lambda(Z_n) \right) dt.$$

Therefore, using the expression of  $\phi'_\lambda$  and  $\phi''_\lambda$  stated in Fact (5.2), we get

$$\mathbf{E}[\theta_n \mid \mathcal{F}_n] = \int_0^1 (1-t) (I_n^1(t) + I_n^2(t)) dt, \quad (5.11)$$

with

$$\begin{aligned} I_n^1(t) &\stackrel{\text{def}}{=} \lambda \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \left( F_\nu(\sqrt{\lambda} V_{n,t}) - F_\nu(\sqrt{\lambda} Z_n) \right) \mid \mathcal{F}_n \right], \\ I_n^2(t) &\stackrel{\text{def}}{=} -\alpha \lambda^\nu \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \left( V_{n,t}^{\alpha-1} F_{1-\nu}(\sqrt{\lambda} V_{n,t}) - Z_n^{\alpha-1} F_{1-\nu}(\sqrt{\lambda} Z_n) \right) \mid \mathcal{F}_n \right]. \end{aligned}$$



Recall that we want to estimate

$$\begin{aligned} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \mu_4(n) \right] &= \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^1(t) dt \right] \\ &\quad + \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^2(t) dt \right]. \end{aligned}$$

We deal with each term separately.

**Dealing with  $I^1$ :** We prove that the contribution of this term is negligible, *i.e.*

$$\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^1(t) dt \right] \right| \leq c_8 \lambda^{\nu+\varepsilon}. \quad (5.12)$$

To this end, we first notice that

$$\begin{aligned} |I_n^1(t)| &\leq \lambda^{\frac{3}{2}} \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \in [Z_n, Z_{n+1}]} |F'_\nu(\sqrt{\lambda}x)| \mid \mathcal{F}_n \right] \\ &= \lambda^{\frac{3}{2}} \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \in [Z_n, Z_{n+1}]} (\sqrt{\lambda}x)^\alpha F_{1-\nu}(\sqrt{\lambda}x) \mid \mathcal{F}_n \right] \\ &\leq c_{1-\nu} \lambda^{\frac{3}{2}} \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \in [Z_n, Z_{n+1}]} (\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x} \mid \mathcal{F}_n \right], \end{aligned} \quad (5.13)$$

where we used (5.2) to find  $c_{1-\nu}$  such that  $F_{1-\nu}(x) \leq c_{1-\nu} e^{-x}$ . We now split (5.13) according to whether

$$(a) \frac{1}{2} Z_n \leq Z_{n+1} \leq 2Z_n \quad \text{or} \quad (b) Z_{n+1} < \frac{1}{2} Z_n \text{ or } Z_{n+1} > 2Z_n.$$

On the one hand, Lemma 4.3 states that

$$\mathbf{E} [|Z_{n+1} - Z_n|^p \mid \mathcal{F}_n] \leq D_p Z_n^{\frac{p}{2}} \quad \text{for all } p \in \mathbb{N} \text{ and } Z_n \neq 0.$$

Hence, for  $1 \leq n \leq \sigma - 1$ , we get

$$\begin{aligned} \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \in [Z_n, Z_{n+1}]} (\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x} \mathbb{1}_{\{\frac{1}{2}Z_n \leq Z_{n+1} \leq 2Z_n\}} \mid \mathcal{F}_n \right] \\ \leq \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \in [\frac{1}{2}Z_n, 2Z_n]} (\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x} \mathbb{1}_{\{\frac{1}{2}Z_n \leq Z_{n+1} \leq 2Z_n\}} \mid \mathcal{F}_n \right] \\ \leq \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 (2\sqrt{\lambda}Z_n)^\alpha e^{-\frac{1}{2}\sqrt{\lambda}Z_n} \mid \mathcal{F}_n \right] \\ \leq c_9 Z_n^{\frac{3}{2}} (\sqrt{\lambda}Z_n)^\alpha e^{-\frac{1}{2}\sqrt{\lambda}Z_n} \\ \leq c_9 \lambda^{\frac{3\alpha-6}{8}} Z_n^{\frac{3\alpha}{4}} (\sqrt{\lambda}Z_n)^{\frac{6+\alpha}{4}} e^{-\frac{1}{2}\sqrt{\lambda}Z_n} \\ \leq c_{10} \lambda^{\frac{3\alpha-6}{8}} Z_n^{\frac{3\alpha}{4}}, \end{aligned} \quad (5.14)$$

where we used the fact that the function  $x^{\frac{6+\alpha}{4}} e^{-\frac{x}{2}}$  is bounded on  $\mathbb{R}_+$  for the last inequality. On the other hand,

$$\begin{aligned}
& \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \in [Z_n, Z_{n+1}]} (\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x} \mathbb{1}_{\{Z_{n+1} < \frac{1}{2}Z_n \text{ or } Z_{n+1} > 2Z_n\}} \mid \mathcal{F}_n \right] \\
& \leq \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \geq 0} (\sqrt{\lambda}x)^\alpha e^{-\sqrt{\lambda}x} \mathbb{1}_{\{Z_{n+1} < \frac{1}{2}Z_n \text{ or } Z_{n+1} > 2Z_n\}} \mid \mathcal{F}_n \right] \\
& \leq c_{11} \mathbf{E} [|Z_{n+1} - Z_n|^6 \mid \mathcal{F}_n]^{1/2} \mathbf{P} \left\{ Z_{n+1} < \frac{1}{2}Z_n \text{ or } Z_{n+1} > 2Z_n \mid \mathcal{F}_n \right\}^{\frac{1}{2}} \quad (5.15) \\
& \leq c_{12} Z_n^{\frac{3}{2}} e^{-\frac{K_2}{2}Z_n} \\
& \leq c_{13}.
\end{aligned}$$

Combining (5.13), (5.14) and (5.15), we get

$$|I_n^1(t)| \leq c_{1-\nu} c_{13} \lambda^{\frac{3}{2}} + c_{1-\nu} c_{10} \lambda^{\frac{3\alpha+6}{8}} Z_n^{\frac{3\alpha}{4}} \leq c_{14} \lambda^{\nu+\frac{2-\alpha}{8}} Z_n^{\frac{3\alpha}{4}}.$$

And therefore

$$\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^1(t) dt \right] \right| \leq c_{14} \lambda^{\nu+\frac{2-\alpha}{8}} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\frac{3\alpha}{4}} \right].$$

Corollary 2.3 states that  $\mathbf{E}[\sum_{n=1}^{\sigma-1} Z_n^{\frac{3\alpha}{4}}]$  is finite so the proof of (5.12) is complete.

**Dealing with  $I^2$ :** It remains to prove that

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^2(t) dt \right] = C_4 \lambda^\nu + o(\lambda^{\nu+\varepsilon}). \quad (5.16)$$

To this end, we write

$$I_n^2(t) = -\alpha \lambda^\nu (J_n^1(t) + J_n^2(t) + J_n^3(t)), \quad (5.17)$$

with

$$\begin{aligned}
J_n^1(t) & \stackrel{\text{def}}{=} \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (F_{1-\nu}(\sqrt{\lambda}V_{n,t})) - F_{1-\nu}(\sqrt{\lambda}Z_n) Z_n^{\alpha-1} \mid \mathcal{F}_n \right], \\
J_n^2(t) & \stackrel{\text{def}}{=} \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}) (F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(0)) \mid \mathcal{F}_n \right], \\
J_n^3(t) & \stackrel{\text{def}}{=} F_{1-\nu}(0) \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}) \mid \mathcal{F}_n \right].
\end{aligned}$$

Again, we shall study each term separately. In view of (5.16) and (5.17), the proof of (e) of Lemma 4.4, when  $\alpha < 1$ , will finally be complete once we established the following three estimates:

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) J_n^1(t) dt \right] = \mathcal{O}(\lambda^{\frac{1-\alpha}{4}}), \quad (5.18)$$

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) J_n^2(t) dt \right] = o(\lambda^\varepsilon), \quad (5.19)$$

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) J_n^3(t) dt \right] = C + o(\lambda^\varepsilon). \quad (5.20)$$

**Proof of (5.18):** Using a technique similar to that used for  $I^1$ , we split  $J^1$  into three different terms according to whether

$$(a) \frac{1}{2}Z_n \leq Z_{n+1} \quad (b) 1 \leq Z_{n+1} < \frac{1}{2}Z_n \quad (c) Z_{n+1} = 0.$$

For the first case (a), we write, for  $1 \leq n \leq \sigma - 1$ , recalling that  $V_{n,t} \in [Z_n, Z_{n+1}]$ ,

$$\begin{aligned} & \left| \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \left( F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(\sqrt{\lambda}Z_n) \right) Z_n^{\alpha-1} \mathbb{1}_{\{\frac{1}{2}Z_n \leq Z_{n+1}\}} \mid \mathcal{F}_n \right] \right| \\ & \leq \lambda^{\frac{1}{2}} \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 Z_n^{\alpha-1} \max_{x \geq \frac{1}{2}Z_n} |F'_{1-\nu}(\sqrt{\lambda}x)| \mid \mathcal{F}_n \right] \\ & = \lambda^{\frac{1}{2}} \mathbf{E} [|Z_{n+1} - Z_n|^3 \mid \mathcal{F}_n] Z_n^{\alpha-1} \max_{x \geq \frac{1}{2}Z_n} \left( (\sqrt{\lambda}x)^{-\alpha} F_{\nu}(\sqrt{\lambda}x) \right) \\ & \leq c_{15} \lambda^{\frac{1}{2}} \mathbf{E} [|Z_{n+1} - Z_n|^3 \mid \mathcal{F}_n] Z_n^{\alpha-1} \max_{x \geq \frac{1}{2}Z_n} \left( (\sqrt{\lambda}x)^{-\alpha} e^{-\sqrt{\lambda}x} \right) \\ & = c_{15} \lambda^{\frac{1}{2}} \mathbf{E} [|Z_{n+1} - Z_n|^3 \mid \mathcal{F}_n] Z_n^{-1} \left( \frac{1}{2} \sqrt{\lambda} \right)^{-\alpha} e^{-\frac{1}{2} \sqrt{\lambda} Z_n} \\ & \leq c_{16} Z_n^{\frac{1}{2}} \lambda^{\frac{1-\alpha}{2}} e^{-\frac{1}{2} \sqrt{\lambda} Z_n} \\ & = c_{16} \lambda^{\frac{1-\alpha}{4}} Z_n^{\frac{\alpha}{2}} \left( (\sqrt{\lambda} Z_n)^{\frac{1-\alpha}{2}} e^{-\frac{1}{2} \sqrt{\lambda} Z_n} \right) \\ & \leq c_{17} \lambda^{\frac{1-\alpha}{4}} Z_n^{\frac{\alpha}{2}}, \end{aligned} \tag{5.21}$$

where we used Lemma 4.3 to get an upper bound for the conditional expectation. For the second case (b), keeping in mind Lemma 5.4, we get

$$\begin{aligned} & \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \left( F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(\sqrt{\lambda}Z_n) \right) Z_n^{\alpha-1} \mathbb{1}_{\{1 \leq Z_{n+1} < \frac{1}{2}Z_n\}} \mid \mathcal{F}_n \right] \\ & \leq c_{18} \lambda^{\frac{1}{2}} \mathbf{E} [|Z_{n+1} - Z_n|^3 Z_n^{\alpha-1} \mathbb{1}_{\{1 \leq Z_{n+1} < \frac{1}{2}Z_n\}} \mid \mathcal{F}_n] \max_{x \geq 1} \left( (\sqrt{\lambda}x)^{-\alpha} e^{-\sqrt{\lambda}x} \right) \\ & \leq c_{19} \lambda^{\frac{1}{2}} \mathbf{E} \left[ Z_n^{\alpha+2} \mathbb{1}_{\{1 \leq Z_{n+1} < \frac{1}{2}Z_n\}} \mid \mathcal{F}_n \right] \lambda^{-\frac{\alpha}{2}} \\ & \leq c_{19} \lambda^{\frac{1-\alpha}{2}} Z_n^{\alpha+2} \mathbf{P} \{ Z_{n+1} < \frac{1}{2}Z_n \mid \mathcal{F}_n \} \\ & \leq c_{19} K_1 \lambda^{\frac{1-\alpha}{2}} Z_n^{\alpha+2} e^{-K_2 Z_n} \\ & \leq c_{20} \lambda^{\frac{1-\alpha}{2}}. \end{aligned} \tag{5.22}$$

For the last case (c), we note that when  $Z_{n+1} = 0$ , then  $V_{n,t} = (1-t)Z_n$ , therefore

$$\begin{aligned} & \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \left( F_{1-\nu}(\sqrt{\lambda}V_{n,t}) - F_{1-\nu}(\sqrt{\lambda}Z_n) \right) Z_n^{\alpha-1} \mathbb{1}_{\{Z_{n+1}=0\}} \mid \mathcal{F}_n \right] \\ & = Z_n^2 (F_{1-\nu}(\sqrt{\lambda}(Z_n(1-t))) - F_{1-\nu}(\sqrt{\lambda}Z_n)) Z_n^{\alpha-1} \mathbf{P} \{ Z_{n+1} = 0 \mid \mathcal{F}_n \} \\ & \leq c_{21} \lambda^{\frac{1}{2}} Z_n^{2+\alpha} e^{-K_2 Z_n} \max_{x \in [Z_n(1-t), Z_n]} (\sqrt{\lambda}x)^{-\alpha} \\ & \leq c_{21} \lambda^{\frac{1-\alpha}{2}} (1-t)^{-\alpha} Z_n^2 e^{-K_2 Z_n} \\ & \leq c_{22} \lambda^{\frac{1-\alpha}{2}} (1-t)^{-\alpha}. \end{aligned} \tag{5.23}$$

Combining (5.21), (5.22) and (5.23), we deduce that, for  $1 \leq n \leq \sigma - 1$ ,

$$\int_0^1 (1-t) |J_n^1(t)| dt \leq c_{23} \lambda^{\frac{1-\alpha}{4}} Z_n^{\frac{\alpha}{2}}.$$

Moreover, according to Corollary 2.3, we have  $\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\frac{\alpha}{2}} \right] < \infty$ , therefore

$$\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) J_n^1(t) dt \right] \right| \leq \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \int_0^1 (1-t) |J_n^1(t)| dt \right] \leq c_{24} \lambda^{\frac{1-\alpha}{4}} \quad (5.24)$$

which yields (5.18).

**Proof of (5.19):** We write

$$J_n^2(t) = \mathbf{E}[R_n(t) \mid \mathcal{F}_n]$$

with

$$R_n(t) \stackrel{\text{def}}{=} (Z_{n+1} - Z_n)^2 (V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}) \left( F_{1-\nu}(\sqrt{\lambda} V_{n,t}) - F_{1-\nu}(0) \right).$$

Again, we split the expression of  $J^2$  according to four cases:

$$\begin{aligned} J_n^2(t) &= \mathbf{E}[R_n(t) \mathbb{1}_{\{Z_{n+1}=0\}} \mid \mathcal{F}_n] + \mathbf{E}[R_n(t) \mathbb{1}_{\{1 \leq Z_{n+1} < \frac{1}{2} Z_n\}} \mid \mathcal{F}_n] \\ &\quad + \mathbf{E}[R_n(t) \mathbb{1}_{\{\frac{1}{2} Z_n \leq Z_{n+1} \leq 2Z_n\}} \mid \mathcal{F}_n] + \mathbf{E}[R_n(t) \mathbb{1}_{\{Z_{n+1} > 2Z_n\}} \mid \mathcal{F}_n]. \end{aligned} \quad (5.25)$$

We do not detail the cases  $Z_{n+1} = 0$  and  $1 \leq Z_{n+1} < \frac{1}{2} Z_n$  which may be treated by the same method used in (5.22) and (5.23) and yields similar bounds which do not depend on  $Z_n$ :

$$\begin{aligned} \mathbf{E}[R_n(t) \mathbb{1}_{\{Z_{n+1}=0\}} \mid \mathcal{F}_n] &\leq c_{25} \lambda^{\frac{1-\alpha}{2}} (1-t)^{-\alpha} \\ \mathbf{E}[R_n(t) \mathbb{1}_{\{1 \leq Z_{n+1} < \frac{1}{2} Z_n\}} \mid \mathcal{F}_n] &\leq c_{26} \lambda^{\frac{1-\alpha}{2}}. \end{aligned}$$

In particular, the combination of these two estimates gives:

$$\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) \mathbf{E}[R_n(t) \mathbb{1}_{\{Z_{n+1} < \frac{Z_n}{2}\}} \mid \mathcal{F}_n] dt \right] \right| \leq c_{27} \lambda^{\frac{1-\alpha}{2}}. \quad (5.26)$$

In order to deal with the third term on the r.h.s. of (5.25), we write

$$\begin{aligned} &|\mathbf{E}[R_n(t) \mathbb{1}_{\{\frac{1}{2} Z_n \leq Z_{n+1} \leq 2Z_n\}} \mid \mathcal{F}_n]| \\ &= \left| \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}) (F_{1-\nu}(\sqrt{\lambda} V_{n,t}) - F_{1-\nu}(0)) \mathbb{1}_{\{\frac{1}{2} Z_n \leq Z_{n+1} \leq 2Z_n\}} \mid \mathcal{F}_n \right] \right| \\ &\leq c_{28} \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \geq \frac{Z_n}{2}} x^{\alpha-2} \int_0^{2\sqrt{\lambda} Z_n} |F'_{1-\nu}(y)| dy \mid \mathcal{F}_n \right] \\ &\leq c_{29} \mathbf{E} [|Z_{n+1} - Z_n|^3 \mid \mathcal{F}_n] \max_{x \geq \frac{Z_n}{2}} x^{\alpha-2} \int_0^{2\sqrt{\lambda} Z_n} y^{-\alpha} dy \\ &\leq c_{30} \lambda^{\frac{1-\alpha}{2}} Z_n^{\frac{1}{2}}. \end{aligned}$$

According to Corollary 2.3, when  $\frac{1}{2} < \alpha < 1$ , we have  $\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{1/2} \right] < \infty$ . In this case, we get

$$\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) \mathbf{E}[R_n(t) \mathbb{1}_{\{\frac{1}{2}Z_n \leq Z_{n+1} \leq 2Z_n\}} \mid \mathcal{F}_n] dt \right] \right| \leq c_{31} \lambda^{\frac{1-\alpha}{2}}. \quad (5.27)$$

When  $0 < \alpha \leq \frac{1}{2}$ , the function  $x^{\frac{2-3\alpha}{4}} e^{-x}$  is bounded on  $\mathbb{R}_+$ , so

$$\begin{aligned} e^{-\lambda Z_n} \int_0^1 (1-t) |\mathbf{E}[R_n(t) \mathbb{1}_{\{\frac{1}{2}Z_n \leq Z_{n+1} \leq 2Z_n\}} \mid \mathcal{F}_n]| dt &\leq c_{30} \lambda^{\frac{\alpha}{4}} Z_n^{\frac{3\alpha}{4}} (\lambda Z_n)^{\frac{2-3\alpha}{4}} e^{-\lambda Z_n} \\ &\leq c_{32} \lambda^{\frac{\alpha}{4}} Z_n^{\frac{3\alpha}{4}}. \end{aligned}$$

Therefore, when  $\alpha \leq \frac{1}{2}$ , the estimate (5.27) still holds by changing  $\lambda^{\frac{1-\alpha}{2}}$  to  $\lambda^{\frac{\alpha}{4}}$ . Hence, for every  $\alpha \in (0, 1)$ , we can find  $\varepsilon > 0$  such that

$$\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) \mathbf{E}[R_n(t) \mathbb{1}_{\{\frac{1}{2}Z_n \leq Z_{n+1} \leq 2Z_n\}} \mid \mathcal{F}_n] dt \right] \right| \leq c_{33} \lambda^\varepsilon. \quad (5.28)$$

We now give the upper bound for the last term on the r.h.s. of (5.25). We have

$$\begin{aligned} \mathbf{E} \left[ R_n(t) \mathbb{1}_{\{Z_{n+1} \geq 2Z_n\}} \mid \mathcal{F}_n \right] &= \mathbf{E} \left[ R_n(t) \mathbb{1}_{\{2Z_n \leq Z_{n+1} \leq \lambda^{-\frac{1}{4}}\}} \mid \mathcal{F}_n \right] \\ &\quad + \mathbf{E} \left[ R_n(t) \mathbb{1}_{\{Z_{n+1} > \max(\lambda^{-\frac{1}{4}}, 2Z_n)\}} \mid \mathcal{F}_n \right]. \end{aligned}$$

On the one hand, when  $Z_n \neq 0$  and  $Z_{n+1} \neq 0$ , we have  $|V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}| \leq 2$  thus, for  $1 \leq n \leq \sigma - 1$ ,

$$\begin{aligned} &\left| \mathbf{E} \left[ R_n(t) \mathbb{1}_{\{2Z_n \leq Z_{n+1} \leq \lambda^{-\frac{1}{4}}\}} \mid \mathcal{F}_n \right] \right| \\ &= \left| \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \left( V_{n,t}^{\alpha-1} - Z_n^{\alpha-1} \right) \left( F_{1-\nu}(\sqrt{\lambda} V_{n,t}) - F_{1-\nu}(0) \right) \mathbb{1}_{\{2Z_n < Z_{n+1} \leq \lambda^{-\frac{1}{4}}\}} \mid \mathcal{F}_n \right] \right| \\ &\leq 2 \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \int_0^{\sqrt{\lambda} Z_{n+1}} x^{-\alpha} F_\nu(x) dx \mathbb{1}_{\{2Z_n < Z_{n+1} \leq \lambda^{-\frac{1}{4}}\}} \mid \mathcal{F}_n \right] \\ &\leq c_{34} \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \int_0^{\lambda^{\frac{1}{4}}} x^{-\alpha} dx \mathbb{1}_{\{Z_{n+1} > 2Z_n\}} \mid \mathcal{F}_n \right] \\ &\leq c_{35} \lambda^{\frac{1-\alpha}{4}} \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \mathbb{1}_{\{Z_{n+1} > 2Z_n\}} \mid \mathcal{F}_n \right] \\ &\leq c_{35} \lambda^{\frac{1-\alpha}{4}} \mathbf{E} \left[ (Z_{n+1} - Z_n)^4 \mid \mathcal{F}_n \right]^{\frac{1}{2}} \mathbf{P} \left\{ Z_{n+1} > 2Z_n \mid \mathcal{F}_n \right\}^{\frac{1}{2}} \\ &\leq c_{36} \lambda^{\frac{1-\alpha}{4}}, \end{aligned}$$

where we used Lemma 4.3 and Lemma 5.4 for the last inequality. On the other hand,

$$\begin{aligned}
& \mathbf{E} \left[ R_n(t) \mathbb{1}_{\{Z_{n+1} > \max(\lambda^{-\frac{1}{4}}, 2Z_n)\}} \mid \mathcal{F}_n \right] \\
&= \left| \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}) (F_{1-\nu}(\sqrt{\lambda} V_{n,t}) - F_{1-\nu}(0)) \mathbb{1}_{\{Z_{n+1} > \max(\lambda^{-\frac{1}{4}}, 2Z_n)\}} \mid \mathcal{F}_n \right] \right| \\
&\leq 2 \|F_{1-\nu}\|_{\infty} \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \mathbb{1}_{\{Z_{n+1} > \max(\lambda^{-\frac{1}{4}}, 2Z_n)\}} \mid \mathcal{F}_n \right] \\
&\leq c_{37} \mathbf{E} \left[ (Z_{n+1} - Z_n)^4 \mathbb{1}_{\{Z_{n+1} > 2Z_n\}} \mid \mathcal{F}_n \right]^{\frac{1}{2}} \mathbf{P} \{ Z_{n+1} > \lambda^{-\frac{1}{4}} \mid \mathcal{F}_n \}^{\frac{1}{2}} \\
&\leq c_{38} Z_n e^{-\frac{K_2}{4} Z_n} \mathbf{P} \{ Z_{n+1} > \lambda^{-\frac{1}{4}} \mid \mathcal{F}_n \}^{\frac{1}{2}} \\
&\leq c_{38} Z_n e^{-\frac{K_2}{4} Z_n} \mathbf{E} [Z_{n+1} \mid \mathcal{F}_n]^{\frac{1}{2}} \lambda^{\frac{1}{8}} \\
&\leq c_{39} \lambda^{\frac{1}{8}}.
\end{aligned}$$

These two bounds yield

$$\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) \mathbf{E} [R_n(t) \mathbb{1}_{\{\frac{1}{2} Z_n \leq Z_{n+1} > 2Z_n\}} \mid \mathcal{F}_n] dt \right] \right| \leq c_{40} \lambda^{\beta} \quad (5.29)$$

with  $\beta = \min(\frac{1-\alpha}{4}, \frac{1}{8})$ . Combining (5.26), (5.28) and (5.29), we finally obtain (5.19).

**Proof of (5.20):** Recall that

$$J_n^3(t) \stackrel{\text{def}}{=} F_{1-\nu}(0) \mathbf{E} [(Z_{n+1} - Z_n)^2 (V_{n,t}^{\alpha-1} - Z_n^{\alpha-1}) \mid \mathcal{F}_n].$$

In particular,  $J_n^3(t)$  does not depend on  $\lambda$ . We want to show that there exist  $C \in \mathbb{R}$  and  $\varepsilon > 0$  such that

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) J_n^2(t) dt \right] = C + o(\lambda^{\varepsilon}). \quad (5.30)$$

We must first check that

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \int_0^1 (1-t) |J_n^2(t)| dt \right] < \infty.$$

This may be done, using the same method as before by distinguishing three cases:

$$\text{(a) } Z_{n+1} \geq \frac{1}{2} Z_n \quad \text{(b) } 1 \leq Z_{n+1} < \frac{1}{2} Z_n \quad \text{(c) } Z_{n+1} = 0.$$

Since the arguments are very similar to those provided above, we feel free to skip the details. We find, for  $1 \leq n \leq \sigma - 1$ ,

$$\int_0^1 (1-t) |J_n^2(t)| dt \leq c_{41} Z_n^{\alpha-\frac{1}{2}} + c_{42} \leq c_{43} Z_n^{\frac{\alpha}{2}}.$$

Since  $\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\frac{\alpha}{2}} \right] < \infty$ , with the help of the dominated convergence theorem, we get

$$\lim_{\lambda \rightarrow 0} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) J_n^2(t) dt \right] = \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \int_0^1 (1-t) J_n^2(t) dt \right] \stackrel{\text{def}}{=} C \in \mathbb{R}.$$

Furthermore we have

$$\begin{aligned} \left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) J_n^2(t) dt \right] - C \right| &= \left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} (1 - e^{-\lambda \sum_{k=0}^n Z_k}) \int_0^1 (1-t) J_n^2(t) dt \right] \right| \\ &\leq c_{43} \mathbf{E} \left[ (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}) \sum_{n=1}^{\sigma-1} Z_n^{\frac{\alpha}{2}} \right]. \end{aligned}$$

And using Hölder's inequality, we get

$$\begin{aligned} \mathbf{E} \left[ (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}) \sum_{n=1}^{\sigma-1} Z_n^{\frac{\alpha}{2}} \right] &\leq \mathbf{E} \left[ (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}) \sigma^{\frac{1}{3}} \left( \sum_{n=1}^{\sigma-1} Z_n^{\frac{3\alpha}{4}} \right)^{\frac{2}{3}} \right] \\ &\leq \mathbf{E} \left[ (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k})^3 \sigma \right]^{\frac{1}{3}} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^{\frac{3\alpha}{4}} \right]^{\frac{2}{3}} \\ &\leq c_{44} \mathbf{E} \left[ (1 - e^{-\lambda \sum_{k=0}^{\sigma-1} Z_k}) \sigma \right]^{\frac{1}{3}} \\ &\leq c_{45} \lambda^\varepsilon \end{aligned}$$

where we used Lemma 5.3 for the last inequality. This yields (5.20) and completes, at last, the proof of (e) of Lemma 4.4 when  $\alpha \in (0, 1)$ .

### 5.2.6 Proof of Lemma 4.4 when $\alpha = 1$

The proof of the lemma when  $\alpha = 1$  is quite similar to the one for  $\alpha < 1$ . Giving a complete proof would be quite lengthy and redundant. We shall therefore provide only the arguments which differ from the case  $\alpha < 1$ .

For  $\alpha = 1$ , the main difference from the previous case comes from the fact that the function  $F_{1-\nu} = F_0$  is not bounded near 0 anymore, a property that was extensively used in the course of the proof when  $\alpha < 1$ . To overcome this new difficulty, we introduce the function  $G$  defined by

$$G(x) \stackrel{\text{def}}{=} F_0(x) + F_1(x) \log x \quad \text{for } x > 0. \quad (5.31)$$

Using the properties of  $F_0$  and  $F_1$  stated in section 5.1, we easily check that the function  $G$  satisfies

- (1)  $G(0) \stackrel{\text{def}}{=} \lim_{x \rightarrow 0^+} G(x) = \log(2) - \gamma$  (where  $\gamma$  denotes Euler's constant).
- (2) There exists  $c_G > 0$  such that  $G(x) \leq c_G e^{-x}$  for all  $x \geq 0$ .
- (3)  $G'(x) = -x F_0(x) \log x$ , so  $G'(0) = 0$ .
- (4) There exists  $c_{G'} > 0$  such that  $|G'(x)| \leq c_{G'} \sqrt{x} e^{-x/2}$  for all  $x \geq 0$ .

Thus, each time we encounter  $F_0(x)$  in the study of  $\mu_k(n)$ , we will write  $G(x) - F_1(x) \log x$  instead. Let us also notice that  $F_1$  and  $F_1'$  are also bounded on  $[0, \infty)$ .

We now point out, for each assertion (a) - (e) of Lemma 4.4, the modification required to handle the case  $\alpha = 1$ .

**Assertion (a):**  $\mathbf{E}[\mu(0)] = C_0\lambda \log \lambda + C'_0\lambda + o(\lambda)$

As in section 5.2.1, we have

$$\begin{aligned}\mathbf{E}[\mu(0)] &= \lambda \mathbf{E} \left[ \int_0^{Z_1} x F_0(\sqrt{\lambda}x) dx \right] \\ &= \lambda \mathbf{E} \left[ \int_0^{Z_1} x G(\sqrt{\lambda}x) dx \right] - \lambda \mathbf{E} \left[ \int_0^{Z_1} x F_1(\sqrt{\lambda}x) \log(\sqrt{\lambda}x) dx \right] \\ &= \lambda \mathbf{E} \left[ \int_0^{Z_1} x \left( G(\sqrt{\lambda}x) - F_1(\sqrt{\lambda}x) \log x \right) dx \right] - \frac{1}{2} \lambda \log \lambda \mathbf{E} \left[ \int_0^{Z_1} x F_1(\sqrt{\lambda}x) dx \right]\end{aligned}$$

and by dominated convergence,

$$\lim_{\lambda \rightarrow 0} \mathbf{E} \left[ \int_0^{Z_1} x \left( G(\sqrt{\lambda}x) - F_1(\sqrt{\lambda}x) \log x \right) dx \right] = \mathbf{E} \left[ \int_0^{Z_1} x \left( G(0) - F_1(0) \log x \right) dx \right].$$

Furthermore, using the fact that  $F'_1$  is bounded, we get

$$\mathbf{E} \left[ \int_0^{Z_1} x F_1(\sqrt{\lambda}x) dx \right] = \frac{F_1(0)}{2} \mathbf{E}[Z_1^2] + \mathcal{O}(\sqrt{\lambda})$$

so that

$$\mathbf{E}[\mu(0)] = C_0\lambda \log \lambda + C'_0\lambda + o(\lambda).$$

**Assertion (b):**  $\mathbf{E}[\sum_{n=1}^{\sigma-1} \mu_1(n)] = o(\lambda)$

This result is the same as when  $\alpha < 1$ , the only difference being that now

$$\mathbf{P}\{Z_\infty > x\} \underset{x \rightarrow \infty}{\sim} \frac{C \log x}{x}.$$

Thus, equality (5.6) becomes

$$\lambda^2 \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n^2 \mathbb{1}_{\{Z_n \leq \frac{-2 \log \lambda}{\sqrt{\lambda}}\}} \right] \underset{\lambda \rightarrow 0^+}{\sim} c_{46} \lambda^{\frac{3}{2}} |\log \lambda|^2$$

and the same upper bound holds.

**Assertion (c):**  $\mathbf{E}[\sum_{n=1}^{\sigma-1} \mu_2(n)] = C_2\lambda \log \lambda + C'_2\lambda + o(\lambda)$

Using the definition of  $G$ , we now have

$$\begin{aligned}\mu_2(n) &= \lambda Z_n F_0(\sqrt{\lambda}Z_n) f_1(Z_n) e^{-\lambda \sum_{k=0}^n Z_k} \\ &= \lambda Z_n f_1(Z_n) e^{-\lambda \sum_{k=0}^n Z_k} \left[ \left( G(\sqrt{\lambda}Z_n) - F_1(\sqrt{\lambda}Z_n) \log(Z_n) \right) - \frac{1}{2} \log \lambda F_1(\sqrt{\lambda}Z_n) \right].\end{aligned}$$

Since  $f_1(x)$  is equal to 0 for  $x \geq M - 1$ , we get the following (finite) limit

$$\begin{aligned}\lim_{\lambda \rightarrow 0} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n f_1(Z_n) e^{-\lambda \sum_{k=0}^n Z_k} \left( G(\sqrt{\lambda}Z_n) - F_1(\sqrt{\lambda}Z_n) \log(Z_n) \right) \right] = \\ \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n f_1(Z_n) (G(0) - F_1(0) \log(Z_n)) \right].\end{aligned}$$



Using the same idea as in (5.8), using also Lemma 5.3 and the fact that  $F'_1$  is bounded, we deduce that

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n f_1(Z_n) e^{-\lambda \sum_{k=0}^n Z_k} F_1(\sqrt{\lambda} Z_n) \right] = \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} Z_n f_1(Z_n) F_1(0) \right] + o(\lambda^\varepsilon)$$

which completes the proof of the assertion.

**Assertion (d):**  $\mathbf{E}[\sum_{n=1}^{\sigma-1} \mu_3(n)] = C_3 \lambda \log \lambda + C'_3 \lambda + o(\lambda)$

We do not detail the proof of this assertion since it is very similar to the proof of (c).

**Assertion (e):**  $\mathbf{E}[\sum_{n=1}^{\sigma-1} \mu_4(n)] = C'_4 \lambda + o(\lambda)$

It is worth noticing that, when  $\alpha = 1$ , the contribution of this term is negligible compared to (a) (c) (d) and does not affect the value of the constant in Proposition 4.1. This differs from the case  $\alpha < 1$ . Recall that

$$\mu_4(n) = -e^{-\lambda \sum_{k=0}^n Z_k} \mathbf{E}[\theta_n \mid \mathcal{F}_n],$$

where  $\theta_n$  is given by (4.8). Recall also the notation  $V_{n,t} \stackrel{\text{def}}{=} Z_n + t(Z_{n+1} - Z_n)$ . Just as in (5.11), we write

$$\mathbf{E}[\theta_n \mid \mathcal{F}_n] = \int_0^1 (1-t)(I_n^1(t) + I_n^2(t)) dt,$$

with

$$\begin{aligned} I_n^1(t) &\stackrel{\text{def}}{=} \lambda \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \left( F_1(\sqrt{\lambda} V_{n,t}) - F_1(\sqrt{\lambda} Z_n) \right) \mid \mathcal{F}_n \right] \\ I_n^2(t) &\stackrel{\text{def}}{=} -\lambda \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (F_0(\sqrt{\lambda} V_{n,t}) - F_0(\sqrt{\lambda} Z_n)) \mid \mathcal{F}_n \right]. \end{aligned}$$

It is clear that inequality (5.13) still holds *i.e.*

$$|I_n^1(t)| \leq \lambda^{\frac{3}{2}} \mathbf{E} \left[ |Z_{n+1} - Z_n|^3 \max_{x \in [Z_n, Z_{n+1}]} \sqrt{\lambda} x F_0(\sqrt{\lambda} x) \mid \mathcal{F}_n \right].$$

In view of the relation

$$F_0(\sqrt{\lambda} x) = G(\sqrt{\lambda} x) - F_1(\sqrt{\lambda} x) \log x - \frac{1}{2} F_1(\sqrt{\lambda} x) \log \lambda,$$

and with similar techniques to those used in the case  $\alpha < 1$ , we can prove that

$$\left| \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) I_n^1(t) dt \right] \right| \leq c_{47} \lambda^{\frac{9}{8}} |\log \lambda| = o(\lambda). \quad (5.32)$$

It remains to estimate  $I_n^2(t)$  which we now decompose into four terms:

$$I_n^2(t) = -\lambda(\tilde{J}_n^1(t) + \tilde{J}_n^2(t) + \tilde{J}_n^3(t) + \tilde{J}_n^4(t)),$$

with

$$\begin{aligned}
\tilde{J}_n^1(t) &\stackrel{\text{def}}{=} \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (G(\sqrt{\lambda}V_{n,t}) - G(\sqrt{\lambda}Z_n)) \mid \mathcal{F}_n \right] \\
\tilde{J}_n^2(t) &\stackrel{\text{def}}{=} -\frac{1}{2} \log \lambda \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (F_1(\sqrt{\lambda}V_{n,t}) - F_1(\sqrt{\lambda}Z_n)) \mid \mathcal{F}_n \right] \\
\tilde{J}_n^3(t) &\stackrel{\text{def}}{=} -\mathbf{E} \left[ (Z_{n+1} - Z_n)^2 \log Z_n (F_1(\sqrt{\lambda}V_{n,t}) - F_1(\sqrt{\lambda}Z_n)) \mid \mathcal{F}_n \right] \\
\tilde{J}_n^4(t) &\stackrel{\text{def}}{=} -\mathbf{E} \left[ (Z_{n+1} - Z_n)^2 (\log V_{n,t} - \log(Z_n)) F_1(\sqrt{\lambda}V_{n,t}) \mid \mathcal{F}_n \right].
\end{aligned}$$

We can obtain an upper bound of order  $\lambda^\varepsilon$  for  $\tilde{J}_n^1(t)$  by considering again three cases:

$$(1) \frac{1}{2}Z_n < Z_{n+1} < 2Z_n \quad (2) Z_{n+1} \leq \frac{1}{2}Z_n \quad (3) Z_{n+1} \geq 2Z_n.$$

For (1), we use that  $|G'(x)| \leq c_{G'}\sqrt{x}e^{-x/2}$  for all  $x \geq 0$ . We deal with (2) combining Lemma 5.4 and the fact that  $G'$  is bounded. Finally, the case (c) may be treated by similar methods as those used for dealing with  $J_n^2(t)$  in the proof of (e) when  $\alpha < 1$  (i.e. we separate into two terms according to whether  $Z_{n+1} \leq \lambda^{-1/4}$  or not).

Keeping in mind that  $F_1$  is bounded and that  $|F_1'(x)| = xF_0(x) \leq c_{48}\sqrt{x}e^{-x}$ , the same method enables us to deal with  $\tilde{J}_n^2(t)$  and  $\tilde{J}_n^3(t)$ . Combining these estimates, we get

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) \left( \tilde{J}_n^1(t) + \tilde{J}_n^2(t) + \tilde{J}_n^3(t) \right) dt \right] = o(\lambda^\varepsilon).$$

for  $\varepsilon > 0$  small enough. Therefore, it merely remains to prove that

$$\lim_{\lambda \rightarrow 0^+} \mathbf{E} \left[ \sum_{n=1}^{\sigma-1} e^{-\lambda \sum_{k=0}^n Z_k} \int_0^1 (1-t) \tilde{J}_n^4(t) dt \right] \quad (5.33)$$

exists and is finite. In view of the dominated convergence theorem, it suffices to prove that

$$\mathbf{E} \left[ \sum_{n=1}^{\sigma-1} \int_0^1 (1-t) \mathbf{E} \left[ (Z_{n+1} - Z_n)^2 |\log V_{n,t} - \log(Z_n)| \mid \mathcal{F}_n \right] dt \right] < \infty. \quad (5.34)$$

We consider separately the cases  $Z_{n+1} > Z_n$  and  $Z_{n+1} \leq Z_n$ . On the one hand, using the inequality  $\log(1+x) \leq x$ , we get

$$\begin{aligned}
&\mathbf{E} \left[ \mathbb{1}_{\{Z_{n+1} > Z_n\}} (Z_{n+1} - Z_n)^2 |\log V_{n,t} - \log(Z_n)| \mid \mathcal{F}_n \right] \\
&\leq \mathbf{E} \left[ \mathbb{1}_{\{Z_{n+1} > Z_n\}} (Z_{n+1} - Z_n)^2 \log \left( 1 + \frac{t(Z_{n+1} - Z_n)}{Z_n} \right) \mid \mathcal{F}_n \right] \leq t\sqrt{Z_n}.
\end{aligned}$$

On the other hand, we find

$$\begin{aligned}
&\mathbf{E} \left[ \mathbb{1}_{\{Z_{n+1} \leq Z_n\}} (Z_{n+1} - Z_n)^2 |\log V_{n,t} - \log(Z_n)| \mid \mathcal{F}_n \right] \\
&\leq \mathbf{E} \left[ \mathbb{1}_{\{Z_{n+1} \leq Z_n\}} (Z_{n+1} - Z_n)^2 \log \left( 1 + \frac{t(Z_n - Z_{n+1})}{Z_n - t(Z_n - Z_{n+1})} \right) \mid \mathcal{F}_n \right] \leq \frac{t}{1-t} \sqrt{Z_n}.
\end{aligned}$$

Since  $\mathbf{E}[\sum_{n=1}^{\sigma-1} \sqrt{Z_n}]$  is finite, we deduce (5.34) and the proof of assertion (e) is complete.

## 6 Proof of Theorem 1.1

Recall that  $X$  stands for the  $(M, \bar{p})$ -cookie random walk and  $Z$  stands for its associated branching process. We define the sequence of return times  $(\sigma_n)_{n \geq 0}$  by

$$\begin{cases} \sigma_0 & \stackrel{\text{def}}{=} 0, \\ \sigma_{n+1} & \stackrel{\text{def}}{=} \inf\{k > \sigma_n, Z_k = 0\}. \end{cases}$$

In particular,  $\sigma_1 = \sigma$  with the notation of the previous sections. We write

$$\sum_{k=0}^{\sigma_n} Z_k = \sum_{k=\sigma_0}^{\sigma_1-1} Z_k + \dots + \sum_{k=\sigma_{n-1}}^{\sigma_n-1} Z_k.$$

The random variables  $(\sum_{k=\sigma_i}^{\sigma_{i+1}-1} Z_k, i \in \mathbb{N})$  are i.i.d. In view of Proposition 4.1, the characterization of the domains of attraction to a stable law implies

$$\begin{cases} \frac{\sum_{k=0}^{\sigma_n} Z_k}{n^{1/\nu}} & \xrightarrow[n \rightarrow \infty]{\text{law}} \mathcal{S}_\nu & \text{when } \alpha \in (0, 1), \\ \frac{\sum_{k=0}^{\sigma_n} Z_k}{n \log n} & \xrightarrow[n \rightarrow \infty]{\text{prob}} c & \text{when } \alpha = 1. \end{cases} \quad (6.1)$$

where  $\mathcal{S}_\nu$  denotes a positive, strictly stable law with index  $\nu \stackrel{\text{def}}{=} \frac{\alpha+1}{2}$  and where  $c$  is a strictly positive constant. Moreover, the random variables  $(\sigma_{n+1} - \sigma_n, n \in \mathbb{N})$  are i.i.d. with finite expectation  $\mathbf{E}[\sigma]$ , thus

$$\frac{\sigma_n}{n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mathbf{E}[\sigma]. \quad (6.2)$$

The combination of (6.1) and (6.2) easily gives

$$\begin{cases} \frac{\sum_{k=0}^n Z_k}{n^{1/\nu}} & \xrightarrow[n \rightarrow \infty]{\text{law}} \mathbf{E}[\sigma]^{-\frac{1}{\nu}} \mathcal{S}_\nu & \text{when } \alpha \in (0, 1), \\ \frac{\sum_{k=0}^n Z_k}{n \log n} & \xrightarrow[n \rightarrow \infty]{\text{prob}} c \mathbf{E}[\sigma]^{-1} & \text{when } \alpha = 1. \end{cases}$$

Concerning the hitting times of the cookie random walk  $T_n = \inf\{k \geq 0, X_k = n\}$ , making use of Proposition 2.1, we now deduce that

$$\begin{cases} \frac{T_n}{n^{1/\nu}} & \xrightarrow[n \rightarrow \infty]{\text{law}} 2\mathbf{E}[\sigma]^{-\frac{1}{\nu}} \mathcal{S}_\nu & \text{when } \alpha \in (0, 1), \\ \frac{T_n}{n \log n} & \xrightarrow[n \rightarrow \infty]{\text{prob}} 2c\mathbf{E}[\sigma]^{-1} & \text{when } \alpha = 1. \end{cases}$$

Since  $T_n$  is the inverse of  $\sup_{k \leq n} X_k$ , we conclude that

$$\begin{cases} \frac{1}{n^\nu} \sup_{k \leq n} X_k & \xrightarrow[n \rightarrow \infty]{\text{law}} \mathcal{M}_\nu & \text{when } \alpha \in (0, 1), \\ \frac{\log n}{n} \sup_{k \leq n} X_k & \xrightarrow[n \rightarrow \infty]{\text{prob}} C & \text{when } \alpha = 1, \end{cases}$$

where  $C \stackrel{\text{def}}{=} (2c)^{-1} \mathbf{E}[\sigma] > 0$  and  $\mathcal{M}_\nu \stackrel{\text{def}}{=} 2^{-\nu} \mathbf{E}[\sigma] \mathcal{S}_\nu^{-\nu}$  is a Mittag-Leffler random variable with index  $\nu$ . This completes the proof of the theorem for  $\sup_{k \leq n} X_k$ . It remains to prove that this result also holds for  $X_n$  and for  $\inf_{k \geq n} X_k$ . We need the following lemma.

**Lemma 6.1.** *Let  $X$  be a transient cookie random walk. There exists  $f : \mathbb{N} \mapsto \mathbb{R}_+$  with  $\lim_{K \rightarrow +\infty} f(K) = 0$  such that, for every  $n \in \mathbb{N}$ ,*

$$\mathbf{P}\{n - \inf_{i \geq T_n} X_i > K\} \leq f(K).$$

*Proof.* The proof of this lemma is very similar to that of Lemma 4.1 of [3]. For  $n \in \mathbb{N}$ , let  $\omega_{X,n} = (\omega_{X,n}(i, x))_{i \geq 1, x \in \mathbb{Z}}$  denote the random cookie environment at time  $T_n$  "viewed from the particle", *i.e.* the environment obtained at time  $T_n$  and shifted by  $n$ . With this notation,  $\omega_{X,n}(i, x)$  denotes the strength of the  $i^{\text{th}}$  cookies at site  $x$ :

$$\omega_{X,n}(i, x) = \begin{cases} p_j & \text{if } j = i + \#\{0 \leq k < T_n, X_k = x + n\} \leq M, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Since the cookie random walk  $X$  has not visited the half line  $[n, \infty)$  before time  $T_n$ , the cookie environment  $\omega_{X,n}$  on  $[0, \infty)$  is the same as the initial cookie environment, that is, for  $x \geq 0$ ,

$$\omega_{X,n}(i, x) = \begin{cases} p_i & \text{if } 1 \leq i \leq M, \\ \frac{1}{2} & \text{otherwise.} \end{cases} \quad (6.3)$$

Given a cookie environment  $\omega$ , we denote by  $\mathbf{P}_\omega$  a probability under which  $X$  is a cookie random walk starting from 0 in the cookie environment  $\omega$ . Therefore, with these notations,

$$\mathbf{P}\{n - \inf_{i \geq T_n} X_i > K\} \leq \mathbf{E} [\mathbf{P}_{\omega_{X,n}}\{X \text{ visits } -K \text{ at least once}\}]. \quad (6.4)$$

Consider now the deterministic (but non-homogeneous) cookie environment  $\omega_{\bar{p},+}$  obtained from the classical homogeneous  $(M, \bar{p})$  environment by removing all the cookies situated on  $(-\infty, -1]$ :

$$\begin{cases} \omega_{\bar{p},+}(i, x) = \frac{1}{2}, & \text{for all } x < 0 \text{ and } i \geq 1, \\ \omega_{\bar{p},+}(i, x) = p_i, & \text{for all } x \geq 0 \text{ and } i \geq 1 \text{ (with the convention } p_i = \frac{1}{2} \text{ for } i \geq M). \end{cases}$$

According to (6.3), the random cookie environment  $\omega_{X,n}$  is almost surely larger than the environment  $\omega_{\bar{p},+}$  for the canonical partial order, *i.e.*

$$\omega_{X,n}(i, x) \geq \omega_{\bar{p},+}(i, x) \quad \text{for all } i \geq 1, x \in \mathbb{Z}, \text{ almost surely.}$$

The monotonicity result of Zerner stated in Lemma 15 of [15] yields

$$\mathbf{P}_{\omega_{X,n}}\{X \text{ visits } -K \text{ at least once}\} \leq \mathbf{P}_{\omega_{\bar{p},+}}\{X \text{ visits } -K \text{ at least once}\} \quad \text{almost surely.}$$

Combining this with (6.4), we get

$$\mathbf{P}\{n - \inf_{i \geq T_n} X_i > K\} \leq \mathbf{P}_{\omega_{\bar{p},+}}\{X \text{ visits } -K \text{ at least once}\}. \quad (6.5)$$

This upper bound does not depend on  $n$ . Moreover, it is shown in the proof of Lemma 4.1 of [3] that the walk in the cookie environment  $\omega_{\bar{p},+}$  is transient which implies, in particular,

$$\mathbf{P}_{\omega_{\bar{p},+}}\{X \text{ visits } -K \text{ at least once}\} \xrightarrow{K \rightarrow \infty} 0.$$

□

We now complete the proof of Theorem 1.1. Let  $n, r, p \in \mathbb{N}$ , using the equality  $\{T_{r+p} \leq n\} = \{\sup_{k \leq n} X_k \geq r+p\}$ , we get

$$\{\sup_{k \leq n} X_k < r\} \subset \{\inf_{k \geq n} X_k < r\} \subset \{\sup_{k \leq n} X_k < r+p\} \cup \{\inf_{k \geq T_{r+p}} X_k < r\}.$$

Taking the probability of these sets, we obtain

$$\mathbf{P}\{\sup_{k \leq n} X_k < r\} \leq \mathbf{P}\{\inf_{k \geq n} X_k < r\} \leq \mathbf{P}\{\sup_{k \leq n} X_k < r+p\} + \mathbf{P}\{\inf_{k \geq T_{r+p}} X_k < r\}.$$

But, using Lemma 6.1, we have

$$\mathbf{P}\{\inf_{k \geq T_{r+p}} X_k < r\} = \mathbf{P}\{r+p - \inf_{k \geq T_{r+p}} X_k > p\} \leq f(p) \xrightarrow{p \rightarrow \infty} 0.$$

Choosing  $x \geq 0$  and  $r = \lfloor xn^\nu \rfloor$  and  $p = \lfloor \log n \rfloor$ , we get, for  $\alpha < 1$ , as  $n$  tends to infinity

$$\lim_{n \rightarrow \infty} \mathbf{P}\left\{\frac{\inf_{k \geq n} X_k}{n^\nu} < x\right\} = \lim_{n \rightarrow \infty} \mathbf{P}\left\{\frac{\sup_{k \leq n} X_i}{n^\nu} < x\right\} = \mathbf{P}\{\mathcal{M}_\nu < x\}.$$

Of course, the same method also works when  $\alpha = 1$ . This proves Theorem 1.1 for  $\inf_{k \geq n} X_k$ . Finally, the result for  $X_n$  follows from

$$\inf_{k \geq n} X_k \leq X_n \leq \sup_{k \leq n} X_k.$$

**Acknowledgments.** *The authors wish to thank Yueyun Hu whose help, advices and ideas were essential for the development of this paper.*

## References

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